Approximating center manifolds on a model for a Solar Sail

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Solar Sailing is a proposed form of spacecraft propulsion using large membrane mirrors. The impact of the photons emitted by the Sun on the surface of the sail and its further reflection produce momentum on it. It is well known that a solar sail is an orientable surface and its orientation is defined by two angles, α and δ. Another important parameter is the sails' lightness number β, which measures its effectiveness. In this work we have considered the motion of a solar sail on the Sun – Earth system. We have used as a model the RTBP plus the solar radiation pressure. The equations of motion read as:

$$
\begin{aligned}
\ddot{x} - 2\dot{y} &= x - (1-\mu)\frac{x-\mu}{r_{PS}^3} - \mu\frac{x-\mu+1}{r_{PE}^3} + \beta\frac{1-\mu}{r_{PS}^2}\,\langle \vec{r}_S, \vec{n}\rangle^2 n_x, \\
\ddot{y} + 2\dot{x} &= y - \left(\frac{1-\mu}{r_{PS}^3} + \frac{\mu}{r_{PE}^3}\right) y + \beta\frac{1-\mu}{r_{PS}^2}\,\langle \vec{r}_S, \vec{n}\rangle^2 n_y, \\
\ddot{z} &= -\left(\frac{1-\mu}{r_{PS}^3} + \frac{\mu}{r_{PE}^3}\right) z + \beta\frac{1-\mu}{r_{PS}^2}\,\langle \vec{r}_S, \vec{n}\rangle^2 n_z,
\end{aligned}
\tag{1}
$$

where $\vec{r}_S$ is a unitary vector describing the Sun - line and $\vec{n}$ is the normal vector to the surface of the sail parametrised by the two angles α and δ.

If the radiation pressure is discarded ($\beta = 0$), it is well known [3], that this model has 5 equilibrium points. When the effect of the solar pressure is added, these fixed points are replaced by a 2D family of equilibrium points parametrised by the angles α and δ ([2, 1]). Most of these fixed points are unstable with a pair of real eigenvalues and two pairs of complex eigenvalues. Our goal is to describe the non-linear behaviour around these equilibrium points.

It can be easily seen that by taking $\alpha = \delta = 0$ (i.e., the sail is perpendicular to the Sun direction) the system is Hamiltonian. In this case this system has 5 equilibrium points, called Sub-$L_{1,\dots,5}$, with three of them placed on the Sun – Earth line. In this presentation we will focus on points of the type centre$\times$centre$\times$saddle. In order to describe the dynamics in a relatively big neighbourhood of the fixed points, we have performed the so-called reduction to the centre manifold. For this particular case we can take advantage of the Hamiltonian properties to do this reduction. A first option is based on expanding the initial Hamiltonian around a given fixed point:

$$H = H_2 + H_3 + \ldots + H_n,$$

and performing a partial normal form scheme, uncoupling (up to high order) the hyperbolic directions from the elliptic ones. We have developed an algebraic manipulator (in ANSI C) tailored for this problem, taking into account the particularities of the problem. With this, we have computed the centre manifold up to order 32. To study the dynamics of the Hamiltonian restricted to the approximated centre manifold we fix the level of energy and take a suitable Poincaré section reducing the system to a 2-D phase space. In this way, varying the energy level we obtain a series of plots describing the dynamics contained in (the approximate) centre manifold. As we can see in Fig. 1, for a fixed level of energy (H), each of the frequencies that define the central motion defines a periodic orbit, the planar and vertical Lyapunov orbits. Around the fixed point we also appreciate a hyperbolic family of invariant tory, that appear due to the coupling of the two frequencies. As the level of energy increases the planar Lyapunov orbit changes its stability and the so-called Halo orbits appear.

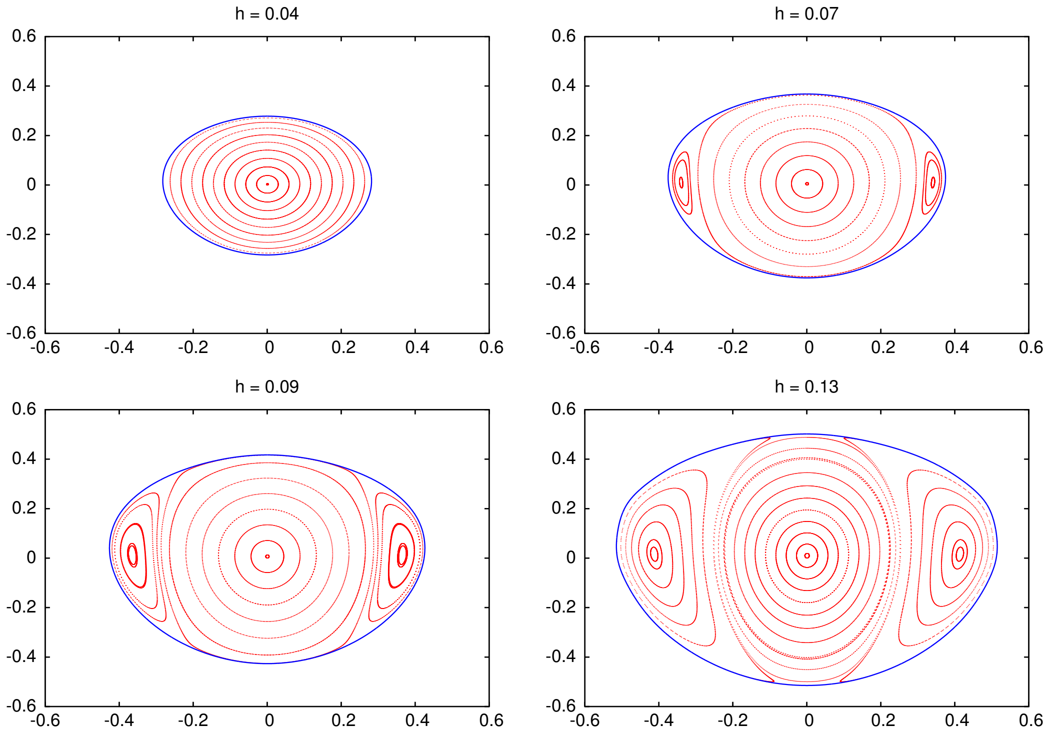

Figure 1: For $\beta = 0.15$, Poincaré section of the centre manifold taking $z = 0$ and a fixed energy value h. From left to right, up and down, $h = 0.04, 0.07, 0.09, 0.13$.

If we take α and δ different from zero, the system is no longer Hamiltonian and we cannot use the same techniques used before. The reduction to the centre manifold must be done on the equations. After some (linear) changes of variables and normalisations, equation (1) can be rewritten as,

$$\begin{aligned} \dot{x} &= Ax + f(x, y), \\ \dot{y} &= By + g(x, y), \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^4$ and $y \in \mathbb{R}^2$. The matrices A and B give the linear approximation to the centre part (A) and to the hyperbolic part (B). The terms f and g denote the nonlinear part. We will look for a function $y = v(x)$, with $v(0) = 0, Dv(0) = 0$, the local expression of the central manifold. It is easy to see that $v(x)$ must satisfy

$$Dv(x)Ax - Bv(x) = g(x, v(x)) - Dv(x)f(x, v(x)). \tag{3}$$

We will represent v as a power series, $v(x) = \sum_{|\alpha|\geq 2} v_\alpha x^\alpha$, $\alpha \in \mathbb{N}^4$, and we will compute the coefficients v_α up to high order. Note that the left hand side of (3) is a linear operator w.r.t $v(x)$ and the right hand side is nonlinear. It can be seen that the coefficients of x^α for order n ($|\alpha| = n$) are defined from the coefficients of orders less than n ($|\alpha| < n$). So, these coefficients can be computed in a recurrent way. As before, we have coded an ANSI C code to find these coefficients in an efficient way up to very high orders. The motion of the flow restricted on the central manifold can be approximated by $\dot{x} = Ax + f(x, v(x))$. We will use this equation to obtain a description of the behaviour in an extended neighbourhood of these equilibrium points.

References

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