On Invariant Manifolds of Conservative Systems with 3-rd and 6-th Degree First Integrals *

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Abstract

One of important characteristics in qualitative analysis of the phase space of mechanical systems, having first integrals, is existence of invariant sets. There is a number of methods for obtaining the invariant sets of mechanical systems, for example, from conditions of symmetry of the problem or stationary conditions for the first integrals of the problem (see [1], [2], for example).

The authors of the paper have for a long time been applying methods of computer algebra in problems of qualitative analysis of mechanical systems. Some results related to investigation of the invariant sets of mechanical systems, which have been obtained with use of computer algebra tools, can be found in [3], [4]. In the present work, an attempt has been made to extend the methods used by the authors in analysis of mechanical systems to dynamic systems in Lie algebras. Such systems, in a number of cases, possess polynomial first integrals of degree higher than 2. Analysis of such systems may be of interest from both mathematical and computational point of view.

Dynamic systems [5], which are described by Euler equations in Lie algebra $so(3,1)$, are considered. Recently, new integrable cases with algebraic integrals of 3-rd and 6-th degree have been obtained for these systems.

For example, the differential equations and the first integrals of the dynamic system, which has the additional cubic first integral, write:

$$
\begin{aligned}
&\dot{M}_1 = M_1(\gamma_3 + 2\alpha_1 M_2) - M_3(\gamma_1 - 4\alpha_3 M_2),\\
&\dot{M}_2 = \gamma_3 M_2 - \gamma_2 M_3 - 2(\alpha_1 M_1 + \alpha_3 M_3)M_1 - 2(\alpha_3 M_1 - \alpha_1 M_3)M_3,\\
&\dot{M}_3 = -2\alpha_1 M_2 M_3,\\
&\dot{\gamma}_1 = 2\alpha_3(\gamma_3 M_2 + \gamma_2 M_3) + (2\alpha_1\gamma_2 + kM_3)M_1 - \gamma_1\gamma_3,\\
&\dot{\gamma}_2 = 2\alpha_1(\gamma_3 M_3 - \gamma_1 M_1) + kM_2 M_3 - \gamma_2\gamma_3 - 2\alpha_3(\gamma_3 M_1 + \gamma_1 M_3),\\
&\dot{\gamma}_3 = \gamma_1^2 + \gamma_2^2 + 2\alpha_3(\gamma_2 M_1 - \gamma_1 M_2) - k(M_1^2 + M_2^2) - 2\alpha_1\gamma_2 M_3.
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
&H = \gamma_2 M_1 - \gamma_1 M_2 - 2M_3(\alpha_1 M_1 + \alpha_3 M_3) + \alpha_3(M_1^2 + M_2^2 + M_3^2) = h,\\
&V_1 = \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 = c_1,\ V_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + k(M_1^2 + M_2^2 + M_3^2) = c_2,\\
&V_3 = M_3(\gamma_1^2 + \gamma_2^2 + \gamma_3^2 - k(M_1^2 + M_2^2 + M_3^2) + 2[\alpha_3(\gamma_2 M_1 - \gamma_1 M_2)\\
&\quad +\alpha_1(\gamma_3 M_2 - \gamma_2 M_3))] = c_3.
\end{aligned}
$$

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The differential equations and the first integrals of the dynamic system, which possesses the additional polynomial 6-th degree first integral, have the form:

$$
\begin{aligned}
&\dot{M}_1 = \frac{1}{2}M_1(2\gamma_3 + \alpha_1 M_2) - M_3(\gamma_1 - \alpha_3 M_2),\\
&\dot{M}_2 = (\gamma_3 M_2 - \gamma_2 M_3) - \frac{1}{2}M_3(\alpha_3 M_1 - \alpha_1 M_3) - \frac{1}{2}(\alpha_1 M_1 - \alpha_3 M_3)M_1,\\
&2\dot{M}_3 = -\alpha_1 M_2 M_3,\\
&\dot{\gamma}_1 = -\gamma_1\gamma_3 + \frac{1}{2}\alpha_1\gamma_2 M_1 + \alpha_3(2\gamma_3 M_2 - \gamma_2 M_3) + kM_1M_3,\\
&\dot{\gamma}_2 = kM_2M_3 - \gamma_2\gamma_3 - \frac{1}{2}\alpha_1(\gamma_1 M_1 - \gamma_3 M_3) - \alpha_3(2\gamma_3 M_1 - \gamma_1 M_3),\\
&\dot{\gamma}_3 = (\gamma_1^2 + \gamma_2^2) + 2\alpha_3(\gamma_2 M_1 - \gamma_1 M_2) - k(M_1^2 + M_2^2) - \frac{1}{2}\alpha_1\gamma_2 M_3.
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
&2\bar{H} = 2(\gamma_2 M_1 - \gamma_1 M_2) - M_3(\alpha_1 M_1 + \alpha_3 M_3) + 2\alpha_3(M_1^2 + M_2^2 + M_3^2) = 2\bar{h},\\
&\bar{V}_1 = \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 = \bar{c}_1,\ \bar{V}_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + k(M_1^2 + M_2^2 + M_3^2) = \bar{c}_2,\\
&\bar{V}_3 = \alpha_1^2 M_3^2\{M_1^2[(\gamma_2 + \alpha_3 M_1)^2 + (\gamma_1 - \alpha_3 M_2)^2] - 2M_1[\alpha_1 M_1(\gamma_2 + \alpha_3 M_1)\\
&\quad +(\gamma_3 + \alpha_1 M_2)(\alpha_3 M_2 - \gamma_1)]M_3 + [(\gamma_2 + 2\alpha_3 M_1)\gamma_2 - kM_1^2\\
&\quad +(\gamma_3 + \alpha_1 M_2)^2]M_3^2 - 2\alpha_1(\gamma_2 + \alpha_3 M_1)M_3^3 + \alpha_1^2 M_3^4\} = \bar{c}_3.
\end{aligned}
$$

Here M_i, γ_i $(i = 1, 2, 3)$ are components of two three-dimensional vectors, α_1, α_2 are some constants, $k = -\alpha_1^2 - \alpha_2^2$.

Systems similar to (1), (2) under some conditions imposed on the parameters of the problem, may be of interest, for example, within the frames of the generalized Kirchhoff model describing motion of a rigid body in ideal fluid, or the Poincare model describing motion of a rigid body with several ellipsoidal cavities filled with vortex fluid [6].

As regards equations (1), (2), problems of obtaining invariant sets and analysis of their properties have been considered. In the capacity of the invariant sets we mean the sets, on which the elements of the family of first integrals assume stationary value. We call such sets as the invariant manifolds of steady motions (IMSMs). In order to obtain these manifolds, stationary conditions of the first integrals of the problem are used.

The analysis of properties IMSMs obtained has been conducted: finding the peculiar invariant manifolds by the method of envelope for the family of the problem's first integrals; investigation of stability by Lyapunov's 2-nd method.

The problem of obtaining "resonance" IMSMs from the following, for example, polynomial relations between the first integrals: $H^3 = \lambda V_3^2$, $(\lambda_0 H - \lambda_1 V_2)^3 = \lambda_2 V_3^2$ has been considered. These relations are valid on desired IMSMs. Investigation of stability for obtained "resonance" IMSMs has been conducted.

Methods of computer algebra (in particular, the method of Gröbner bases) have been applied in the capacity of computational methods. Computer algebra systems *Mathematica* and *Maple* have been used.

References

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