Planetary Three-Body Problem with Poisson Series Processor

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Orbital evolution of a planetary system similar to our Solar one represents one of the most important problems of Celestial Mechanics. The secular behaviour of the planetary three-body problem has been extensively investigated both from the mathematical and numerical point of view by many researchers.

In present work we continue researches of the spatial Planetary Three-Body Problem beginning in [8, 10, 11, 12, 13, 14, 15]. The aim is the investigation of the dynamical evolution of a weakly perturbed spatial Two-Planetary System on a cosmogonic time-scale. This research is based on the presentation of the planetary Hamiltonian as Poisson series with respect to all elements and the use of the Lie transforms method to construct the averaged Hamiltonian up to the third power of small parameter μ. Numerical integration of the averaged equations of motion allows us to study orbital evolution of Jupiter – Saturn system on very-long-time 10 Gyr. Analysis of the variable change functions shows boundaries of validity of this approach with respect to eccentricities, inclinations, semi-major axes and masses of the planets.

We use Jacobian coordinates as the best-fitting. Let us assume m_0, $\widetilde{m}_1 = \mu m_0 m_1$, $\widetilde{m}_2 = \mu m_0 m_2$ as masses of the Sun, Jupiter and Saturn respectively. Small parameter μ can be chosen as $\mu \approx \max(\widetilde{m}_1, \widetilde{m}_2)/m_0$ or $\mu \approx (\widetilde{m}_1 + \widetilde{m}_2)/m_0$. For the system the Sun – Jupiter – Saturn we choose the small parameter μ equal to 10^{-3}. In this case the dimensionless masses m_1 and m_2 are the unity order values ($m_1 \approx 1$, $m_2 \approx 1/3$).

Let us represent the Hamiltonian as a sum

$$h = h_0 + \mu h_1\,, \tag{1}$$

$$h_0 = -\frac{Gm_0m_1}{2a_1} - \frac{Gm_0m_2}{2a_2}\,, \qquad h_1 = \frac{Gm_0}{a_0}h_2\,, \qquad h_2 = h_3 + h_4\,, \tag{2}$$

$$h_3 = \frac{m_2a_0}{\mu}\left(\frac{1}{r_2} - \frac{1}{\rho}\right) = \frac{m_2a_0\left[2\frac{m_1}{1+\mu m_1}\mathbf{r}_1\mathbf{r}_2 + \mu\left(\frac{m_1}{1+\mu m_1}\right)^2 r_1^2\right]}{r_2\rho(r_2+\rho)},$$

$$h_4 = -\frac{m_1m_2a_0}{\Delta}, \quad \rho = \left|\mathbf{r}_2 + \frac{\mu m_1}{1+\mu m_1}\mathbf{r}_1\right|\,, \quad \Delta = \left|\mathbf{r}_2 - \frac{1}{1+\mu m_1}\mathbf{r}_1\right|\,,$$

where G being the gravitational constant, a_0 is the arbitrary parameter with the length dimensionality. Here and below the subscripts 1 and 2 for coordinates and elements correspond to Jupiter and Saturn respectively.

Introduce two systems of osculating elements. The first system is close to the Keplerian one

$$x^{(1)}_{3s-2} = \widetilde{a}_s, \quad x^{(1)}_{3s-1} = e_s, \quad x^{(1)}_{3s} = \widetilde{I}_s, \qquad y^{(1)}_{3s-2} = \alpha_s, \quad y^{(1)}_{3s-1} = \beta_s, \quad y^{(1)}_{3s} = \gamma_s\,. \tag{3}$$

Here $\widetilde{a} = (a - a^0)/a^0$, $\widetilde{I} = \sin(I/2)$, $\alpha = l + g + \Omega$, $\beta = g + \Omega$, $\gamma = \Omega$ are expressed in terms of Keplerian elements a, a^0, e, I, l, g, Ω: semi-major axis and its mean value, eccentricity, inclination, mean anomaly, argument of pericenter, longitude of ascending node; $s = 1, 2$.

The second system realizes simplifications due to the homogeneity of the perturbation function with respect to semi-major axes. In this system denominators arising in a process of averaging transforms are extremely simple. On the other hand, it has a deficiency, mixing several elements of all planets

$$x^{(2)}_{3s-2} = z_s, \quad x^{(2)}_{3s-1} = e_s, \quad x^{(2)}_{3s} = \widetilde{I}_s, \qquad y^{(2)}_{3s-2} = \alpha_s, \quad y^{(2)}_{3s-1} = \beta_s, \quad y^{(2)}_{3s} = \gamma_s \,, \tag{4}$$

where $z_1 = \omega^0_1/\omega_1 - 1$, $z_2 = (\omega^0_1\omega_2)/(\omega^0_2\omega_1) - 1$. Here $\omega_s = \kappa_s a_s^{-3/2}$ are mean motions of the planets, ω^0_s are constants close to mean values ω_s, $\kappa_s^2 = Gm_0m_s/M_s$ are gravitational parameters of the planets, reduced masses are $M_s = m_s(1 + \mu m_1 + \cdots + \mu m_{s-1})/(1 + \mu m_1 + \cdots + \mu m_s)$, $s = 1, 2$.

The disturbing Hamiltonian h_2 is presented as Poisson series

$$h_2 = \sum A_{kn} x^k \cos ny \,. \tag{5}$$

Here $x = \{x_1, \ldots, x_6\}$ are positional elements, $y = \{y_1, \ldots, y_6\}$ are angular ones, A_{kn} are numerical coefficients, $k = \{k_1, \ldots, k_6\}$ and $n = \{n_1, \ldots, n_6\}$ are multi-indices. The summation is taken over non-negative k_s and n_1 and integer $n_2, ..., n_6$.

It is well known that

$$n_1+n_2+n_3+n_4+n_5+n_6 = 0 \,, \quad n_3+n_6 = \text{even} \,, \quad k_s = |n_s| + \text{non-negative even} \;\; (s = 2, 3, 5, 6)$$

The simplest restrictions on k and n are

$$k_1 + k_2 + \ldots + k_6 \le d \,, \qquad n_1 \le c \,, \qquad |n_4| \le c \,. \tag{6}$$

The other n_s are less than d according to D'Alembertian properties of the Hamiltonian h_2 [6, 7].

Designate b the order of approximation with respect to μ. Let us consider system the Sun – Jupiter – Saturn. To evaluate $d(b)$ it is sufficient to take into account that Jupiter's and Saturn's eccentricities $e_1 = 0.05$, $e_2 = 0.05$ and the sines of the half-angles of inclinations $I_1 = 0.01$, $I_2 = 0.02$ are small values of the same order $\mu_1 \sim \mu^{1/2}$. The choice of $c(b)$ is controlled by the rate of convergence at $\mu_1 = 0$. According to [10]:

$$d(2) = 6 \,, \quad d(3) = 11 \,, \quad d(4) = 16, \qquad c(2) = 13 \,, \quad c(3) = 25 \,, \quad c(4) = 37 \,. \tag{7}$$

The parameters a_0, m_1, m_2, a^0_1, a^0_2 (necessary for calculations of the coefficients A_{kn} and the multiplier to the Hamiltonian h_1 transition) are given in [10]. It is shown there that expansions we need up to μ^4 has only one small divisor $2\omega^0_1 - 5\omega^0_2$. The constant $F = |2\omega^0_1 - 5\omega^0_2|/\omega^0_1$ describing the range of commensurability is equal to $0.023331 \sim \sqrt{\mu}$ for chosen values of the parameters. It indicates that we deal with a shallow resonance case.

The Poisson series processor PSP [1, 3] is used to construct the expansion of disturbing Hamiltonian h_2 into Poisson series (5). The rational version of the PSP is used to decrease round-off errors during calculations of the coefficients A_{kn}.

Two variants of the expansion are constructed for the first system of the osculating elements (3). The first variant deals with numerical values of parameters (masses, mean values of semi-major axes, ...) corresponding to the system the Sun – Jupiter – Saturn. The second one deals with their literal expressions depending on parameters of the system. In the last case the parameters are the additional positional variables m_1, m_2, μ, $d_1^2 = 1/a^0_2$, $d_2^2 = a^0_1/(a^0_2(1 + \mu m_1))$ and the corresponding indices are $k_7, \ldots, k_{11}$. The summation is taken over $k_1 + \cdots + k_6 \le 6$, $|n_s| \le 15$ $(s = 1, ..., 6)$, $\quad k_7, k_8 \le 19, \quad k_9 \le 2, \quad k_{10}, k_{11} \le 38$.

The Lie transforms method is used to construct the averaged Hamiltonian H. This method is based on Poisson brackets that allows us to use non-canonical elements writing down the Poisson brackets in the corresponding system of phase variables [9, 14]. The averaged Hamiltonian H, generating function and change of variables functions are echeloned Poisson series [16].

For symbolic operations we use the rational version of the echeloned Poisson series processor EPSP [4] to reduce the round-off errors. The generating function, change of variables functions, and right-hand sides of averaged equations of motion are obtained. For the first system of elements (3) the expansions with numerical parameters are constructed to the Sun – Jupiter – Saturn system upto μ^3. Series with litteral parameters are obtained upto μ^2. For the second system (4) the first approximation is realized only.

The averaged equations are integrated numerically at the time-scale of 10 Gyr. The equations for slow variables are integrated by 15 order Everhart and 11 order Runge-Kutta methods. The equations for fast variables are integrated by spline interpolation method.

An accuracy of the integration is controlled by computation of the integrals of energy and area. The absolute value of the relative error for the energy integral calculation is less than $5.2 \cdot 10^{-13}$ for both integrators at time-scale 10 Gyr. The mean value of the relative error is a constant at all integration time.

The area integrals for initial system in Jacobian coordinates conserve their form [2]. They survive under the averaging transforms [5]. For the accuracy control we used the area integrals referred to the Laplace plane. In this case the vector $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is directed along z-axis. Hence $\sigma_x = \sigma_y = 0$. The absolute value of the relative error for the σ_z achieves $3.5 \cdot 10^{-10}$ for both integrators at time-scale 10 Gyr. The amplitude and mean value of the relative error are constant at all integration time. We find out that the area integrals σ_x, σ_y are preserved with low accuracy: $|\sigma_x/\sigma_{z0}| < 8.8 \cdot 10^{-7}$, $|\sigma_y/\sigma_{z0}| < 5.6 \cdot 10^{-7}$. Non-conservation of σ_x and σ_y integrals has the following reason. As it is proved in [5] they are preserved in the system determined by the averaged Hamiltonian H. But *they are not preserved in a system determined by a finite sequence of Poisson expansion* of averaged Hamiltonian H.

The relative differences between the first and second approximations exceed the small parameter $\mu = 1 \cdot 10^{-3}$, but are less than the quotient $\mu/F = 4.4 \cdot 10^{-2}$.

The short-period perturbations of Jupiter and Saturn semi-major axes a do not exceed 0.0023 a.u. and 0.0122 a.u. respectively. The maximum values of the short-period perturbation norms for the eccentricities e and inclinations I are much less than the amplitudes of the corresponding long-period perturbations. The variable change function norms for the longitudes α, β and γ are much less than the amplitudes of the long-period perturbations also. The variable change functions for the longitudes α, β, γ had no terms independent of trigonometric variables. Hence the variable change functions values are much less than the amplitudes of the corresponding long-period perturbations at all considered cosmogonic time-scale.

The evolution of the two-planetary weakly disturbed Sun–Jupiter–Saturn system has been studied on cosmogonic time scales and quasi-periodicity of the planetary motion has been demonstrated. The eccentricities and inclinations of the orbits of Jupiter and Saturn are retained small but different from zero. The orbital planes and the orbits themselves demonstrate the secular motion. The short-period perturbations are kept small over the entire time period under consideration.

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