Symbolic Creation of Nilpotent Normal Forms: Another Point of View

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ABSTRACT
We begin with a restatement of Poincaré-Dulac's normal form theorem. More precisely, it is shown that a nonlinear system of ordinary differential equations can always be taken to the classical Poincaré-Dulac normal form, but with a number of nonlinear components no more than the number of Jordan blocks of the leading matrix. The approach is couched in the Lie-Poincaré scheme and will be presented as a welcome framework to deal with normal forms of nilpotent systems. Indeed, in a more general context, a number of well-known results on such normal forms can be embedded within the present approach. The computer algebra side will be considered within the frame of the Poincaré Code, a Maple normal form package presented last summer within the Leonhard Euler Congress, held in Saint-Petersburg.

Keywords: Poincaré-Dulac's Normal Form; Nilpotent Leading Matrix; Computer Algebra.

There is two well-known styles for normalizing systems of ordinary differential equations with nilpotent linear part, dealing bona fide with the classical problem of creating a complement to the range of the nilpotent homological operator: the inner-product [1] [4] and the $sl\,(2)$ styles [2], as called by Murdock [5]. Another important result, but regrettably not cited enough in the literature, has been found by Stolovitch [6] when dealing with normal forms via Carleman linearization procedure, thereby generalizing to any dimension a well-known result on the subject by Takens [7].

Our modest contribution is motivated by a recent work of Dullin and Meiss [3] where, after recalling the two commonly used methods, they show that, for a nilpotent divergence-free vector field in $\mathbb{R}^3$ with maximal Jordan block as leading matrix, normal form coordinates can be chosen so that the nonlinear terms occur just as a single function of two variables in the third component. So, we propose a generalization to $\mathbb{R}^n$, thereby improving (and renormalizing) at the same time Stolovitch result. As will be seen, the approach is mainly based on a restatement of the Poincaré-Dulac normal form theorem, a shockingly easy-to-obtain yet very interesting result.

1 Restatement of the Poincaré-Dulac theorem

Let there be a formal nonlinear vector field of $\mathbb{C}^n$ of the form

$$v = Jx + \sum_{k \geq 2} f_k(x)$$

where J is a matrix $n \times n$ (which is, for simplicity's sake, assumed to be in Jordan normal form) and the f_k's are homogeneous vector polynomials of degree k. We will say that a $1 \leq r \leq n$ is *resonant* if the $r-$th row of J_n, the nilpotent part of the matrix J, contains only zero entries. Then, for *any* nonresonant r, we have shown that, by introducing the linear degree-preserving first order differential operator

$$\Upsilon = \left\{ \sum_{i=1}^{n} (Jx)_i \frac{\partial}{\partial x_i} - \lambda_r I \right\}$$

where $(Jx)_i$ is the $i-$th component of Jx, I is the Identity operator and λ_r is the eigenvalue of J such that $(Jx)_r = \lambda_r x_r + x_{r+1}$, we can eliminate *any* monomial $M = x^Q$ ($|Q| \geq 2$) from the $r-$th component of v, but for the smallest resonant s such that $s \geq r$, we have creation of the term

$$M' = \Upsilon^{s-r}(M) = \underbrace{\Upsilon(...(\Upsilon}_{(s-r)\ times}(M)))$$

which will be added to $(vx)_s$. This is done by applying to v a near-identity transformation generated by the following

$$g_M = - \begin{bmatrix} \left.\begin{matrix} 0 \\ | \\ 0 \end{matrix}\right\} (s-r)\ times \\ M \\ \Upsilon(M) \\ | \\ \Upsilon^{s-r-1}(M) \\ 0 \\ | \\ 0 \end{bmatrix}$$

$\exp(g_M)$ being a good candidate. An immediate consequence of this fact is the following:

Proposition 1 *(On a Poincaré-Dulac Theorem)*
Let v be a formal vector field of $\mathbb{C}^n$ which can be written in the neighborhood of the origin as

$$v = \sum_{i=1}^{n} s_i(x) \frac{\partial}{\partial x_i}$$

$s_i \in \mathbb{C}[x]$, the space of formal series in x, with coefficients in $\mathbb{C}$ and without constant terms. We denote by $\lambda_1, ..., \lambda_r$ $(r \leq n)$ the distinct eigenvalues of the linear part, and by $m_1, ..., m_r$ their respective multiplicities.
Then, there exist a formal diffeomorphism H which the action on v is of the form

$$H{*}v = \sum_{i=1}^{r} \left\{ \sum_{j=(m_1+\cdots+m_{i-1})+1}^{(m_1+\cdots m_i)-1} (\lambda_i x_j + x_{j+1}) \frac{\partial}{\partial x_j} + \Big(\sum_{\substack{(Q,\lambda)=\lambda_i \\ Q\in\mathbb{N}^n, |Q|\geq 2}} c_Q^i x^Q \Big) \frac{\partial}{\partial x_{m_1+\cdots+m_i}} \right\}$$

where $m_0 = 0$ as a convention.

As can be seen, resonant monomials (in the sense of Poincaré) appear only in components of the form $(H * v)_{m_1+\cdots+m_i}$, $i = 1, ..., r$. In particular, in the case where the leading matrix is a maximal Jordan block, the result insures the existence of a normalizing transformation H such that $(H * v)_i$ is linear for all $1 \leq i < n$. This is done in a completely algorithmic way.

2 Symbolic creation of nilpotent normal forms: *first level normalization*

J is a nilpotent matrix from now on, and for simplicity's sake, it's assumed to be a maximal nilpotent Jordan block. Obviously, by proposition 1, the vector field v can be transformed by a near-identity transformation into the form

$$Jx + f(x)\, e_n \quad with \quad e_n = (0, ..., 0, 1)^T$$

We are about to generalize to any dimension the result by Dullin and Meiss. It just remains to show the divergence-free character of the obtained normal form, or equivalently, the nondependance of the function f on the last coordinate. For this, recall that for our case, the homological equation is given for any degree $k \geq 2$ by

$$\left[J, \overbrace{g_k}^{?} \right] = f_k$$

where $[.,.]$ is the usual Lie bracket of vector fields and f_k is such that $(f_k)_i = 0$ for all $1 \leq i < n$. Then using the triangular character of J, and "encoding" the homological operator $adJ := [J, .]$ on the usual lexicographic basis B_x, where the coordinates x_i are seen as letters, one can see that monomials that cannot be eliminated are the successors in the basis B_x of monomials of the form

$$x^Q, \quad Q = (q_1, ..., q_{n-2}, 0, q_n) \in \mathbb{N}^n \quad \text{with} \quad q_n \neq 0$$

thereby meeting and improving Stolovitch result obtained via Carleman linearization. The following holds:

Proposition 2
For $n-$dimensional nilpotent systems with a maximal nilpotent Jordan block, a normal form can be chosen as follows (with obvious notations and with the convention $x_{n+1}=0$):

$$H * v = \sum_{i=1}^{n} \left(x_{i+1} + \delta_{i,n} \sum_{Q \in \mathbb{N}^{n-1} \times \{0\},\ |Q| \geq 2} c_Q x^Q \right) \frac{\partial}{\partial x_i}$$

the last nonzero component of Q being always ≥ 2.

The present normal form has already been found by Stolovitch, except the presence of the Kronecker symbol $\delta_{i,n}$, which makes all the difference!

3 Symbolic creation of nilpotent normal forms: *second level normalization*

As an improvement of proposition 2, we show the following

Proposition 3
Monomials of the form x^Q with $Q = (q_1, ..., q_{n-2}, 2, 0)$ (resp. $Q = (q_1, 3)$) can still be eliminated when $n > 3$ (resp. $n = 3$), the $2-$dimensional case corresponding to Takens normal form.

The interested reader can profitably compare the proposed nilpotent normal form with the ones given in:

[1] *Belitskii G., $C^{\infty}-$ normal forms of local vector fields. Acta Appl. Math., 70 (1-3): 23-41, 2002. Symmetry and Perturbation Theory.*

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[4] *Elphick C., Tirapegui E., Brachet M.E., Coullet P. and Iooss G., A simple global characterisation for normal forms of singular vector fields. Phys. D, 29 (1-2): 95-127, 1987.*

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