The study of Markov processes on 3D Schur graph

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Outline

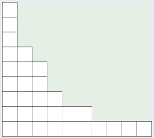
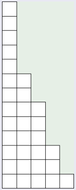
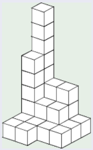
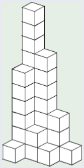
Young diagrams

Definitions

2D Young diagram is a finite collection of boxes, arranged in left-justified rows, with the row lengths in non-increasing order.
3D Young diagram (plane partition) is a two-dimensional array of integers $n_{i,j}$ that are nonincreasing both from left to right and top to bottom and that add up to a given number n.

Young diagrams

Types of diagrams

	Standard	Strict
2D		
3D		

The connection between problems in asymptotic combinatorics and asymptotic representation theory

There exists a one-to-one correspondence between 2D Young diagrams and representations of symmetric group $S(n)$:

The set of **standard** Young diagrams of size n	$\Longleftrightarrow$	The **irreducible** representations of symmetric group $S(n)$
The set of **strict** Young diagrams of size n	$\Longleftrightarrow$	The **projective** representations of symmetric group $S(n)$
The dimension of a diagram	$\Longleftrightarrow$	The dimension of corresponding representation

Graded graphs

Definition

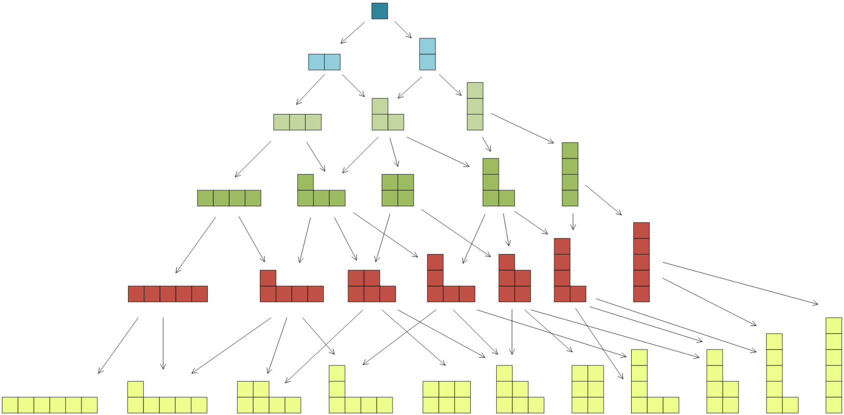
A *graded graph* (or Bratteli-Vershik diagram) is a graph consisting of levels enumerated by integers, where edges connect the vertices of n-th and $(n+1)$-th levels. The word *diagram* in this context means the graded graph itself, but not a single Young diagram.

Examples

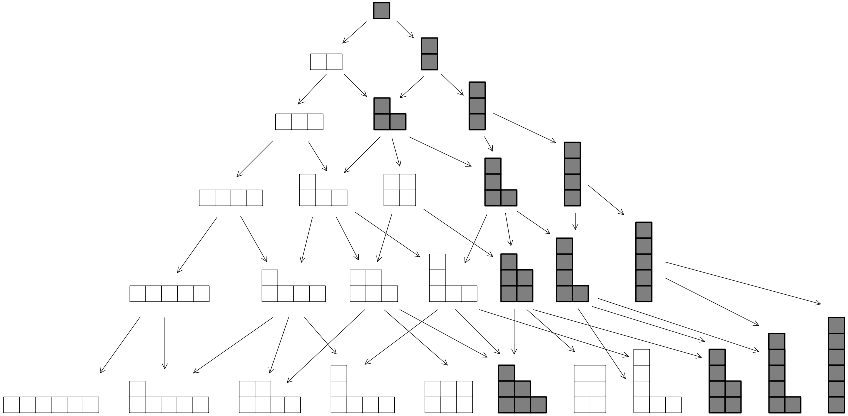
- Pascal graph (2D/3D)
- Young graph (2D/3D)
- Schur graph (2D/3D)

We consider Markov processes with transitions from level n to level $n+1$.

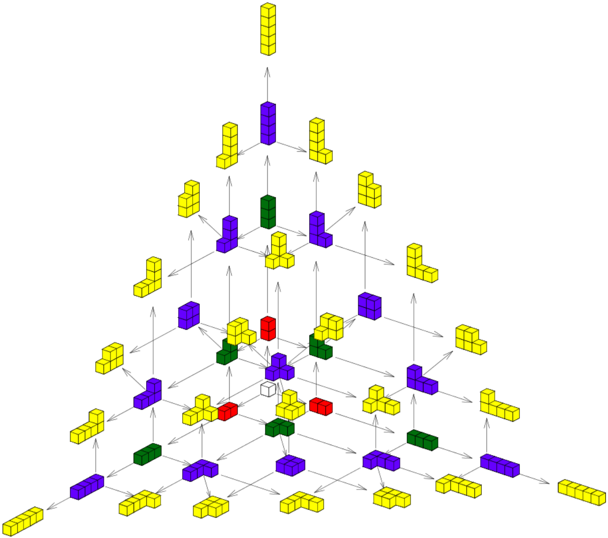
2D Young graph (first 6 levels)



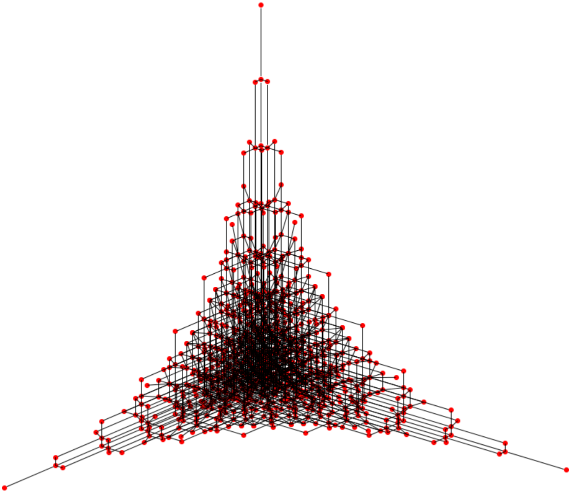
2D Schur graph (first 6 levels)



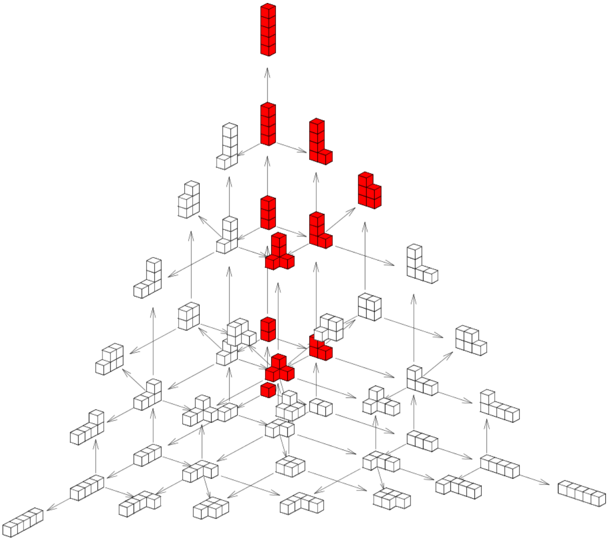
3D Young graph (first 5 levels)



3D Young graph (first 10 levels)



3D Schur graph (first 5 levels)



Central measure

Definition

The dimension of a diagram is the number of paths to this diagram from the root of the corresponding graph.

Definition

The measure is called *central* if all paths between two fixed diagrams have the same probabilities.

2D case:
There is such a Markov process which generates a central measure on the diagrams. It is called *Plancherel process*.

Asymptotic centrality

Unfortunately, there are no known central processes in 3D case.

Definition

Asymptotically central measure* is a probability measure on paths in Young graph, such as

$$\forall k \in \mathbb{N}, \forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \in \mathbb{N} : n > N, \forall \lambda_1 \in Y_n, \forall \lambda_2 \in Y_{n+k}$$

the following inequality is fulfilled:

$$\left| \frac{p_1(\lambda_1, \lambda_2)}{p_2(\lambda_1, \lambda_2)} \right| < 1 + \epsilon,$$

where Y_n - the n-th level of the graph, $p(\lambda_1, \lambda_2)$ - the probability of a path between diagrams λ_1, λ_2

*Vasiliev, N.N., Terentjev, A.B.: Modelling of almost central measures generated by Markov processes in the three-dimensional case. J. Math. Sci. **209**(6), 851–859 (2015)

Greedy paths

Definition

A *greedy path* in a graded graph is a deterministic sequence of diagrams built in the following way: on each step the box with the maximum possible probability is added to the diagram.

Motivation: in 2D case it helps to construct paths to the diagrams with very large dimensions, i.e. it helps to study the asymptotics of growth of dimensions of irreducible representations [Kassel'16].

[VIDEO: greedy growth]

Merging conjecture

A pair of greedy sequences started from an arbitrary pair of diagrams of the same size merge into a single sequence in a finite number of steps.

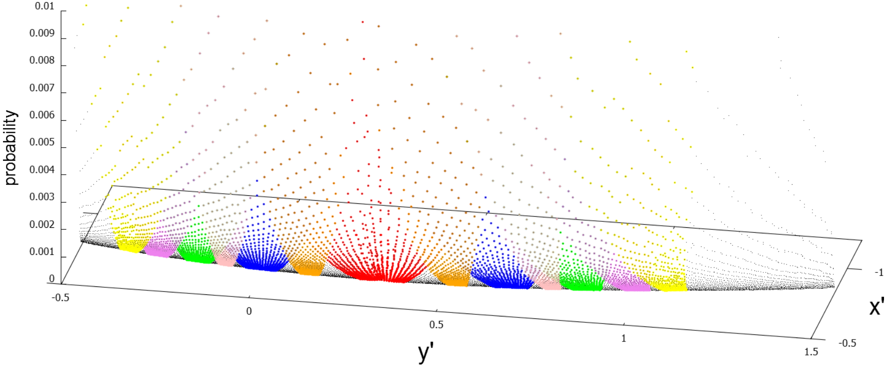
The following papers are devoted to the merging conjecture for different cases:

2D (standard):
N. Vasilyev, V. Duzhin, [Information and Control Systems], 2015
2D (strict):
V. S. Duzhin, N. N. Vasilyev, [J. Knot Theory Ramifications], 2016
3D standard, $\alpha = 1$:
N. N. Vasiliev, V. S. Duzhin, [Zap. Nauchn. Sem. POMI], 2016
3D standard, $\alpha = 0.739$:
V. Duzhin, N. Vasilyev [Mathematics in Computer Science], 2017
3D strict: TBD

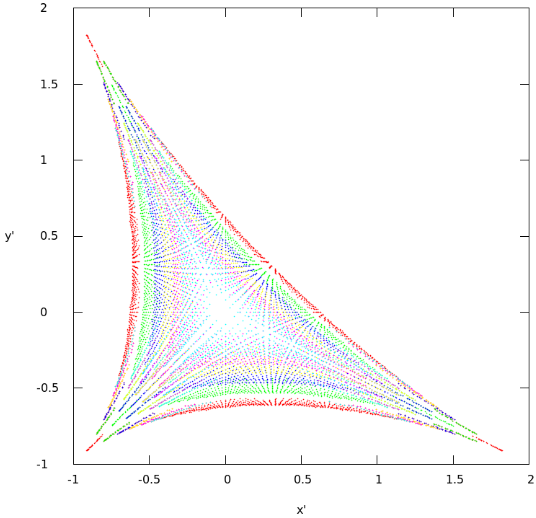
Limit distribution of probabilities

Let us consider a greedy process on 3D Young graph. For each added box we project the corresponding probability on the diagram's front. These projections form the limit distribution of probabilities.

Limit distribution of probabilities: side view



Limit distribution of probabilities: top view



Asymptotically central process on 3D Schur graph

Previously an asymptotically central process on 3D standard graph was studied in [Vasiliev, Terentjev 2015]. Analogously we have constructed the process on Schur graph with the following transition probabilities:

$$f(\lambda, x, y, z) = \prod_{i=0}^{x-1} \frac{h'(\lambda, i, y, z)}{h'(\lambda, i, y, z) + 1} \prod_{j=0}^{y-1} \frac{h'(\lambda, x, j, z)}{h'(\lambda, x, j, z) + 1} \prod_{k=0}^{z-1} \frac{h'(\lambda, x, y, k)}{h'(\lambda, x, y, k) + 1},$$

where h' - hook length of a skewed strict diagram λ, x, y, z are the coordinates of an added box.
This process is called *pseudo-Plancherel* process.

Asymptotically central process on 3D Schur graph

In order to study the properties of asymptotic centrality, we considered weighted paths and parametrized these paths using the vectors of weights of lexicographical orderings.
In Gröbner bases theory such a parametrization of monomial ordering is called the parametrization of Robbiano.

$$v = ax + by + cz,$$

$$a + b + c = 1$$

On each step we add the box (x, y, z) with the maximum v.

Normalized dimension

Definition

Normalized dimension is a numerical parameter which characterizes the asymptotic speed of growth of exact dimension.

Normalized dimension of a 2D diagram was introduced by Vershik in [Vershik-Kerov '82].
We introduce a normalized dimension of a path to a diagram which is connected to one-parameter family of Markov processes.

Normalized dimension

The transition probabilities of some Markov process can be raised to the power α and used as weights to calculate the probabilities of a new process.
Normalized dimension of a path to diagram λ which we introduce for such probabilities is expressed by the formula:

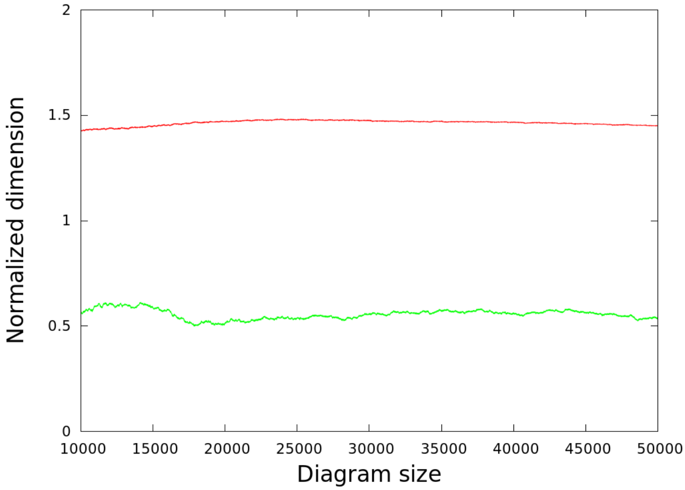
$$c(\lambda) = -\frac{\sum_{i=1}^{n} \ln p_i(\alpha)}{\sqrt[3]{n^2}} - \frac{2}{3}\sqrt[3]{n} \cdot \ln n + C(\alpha) \cdot \sqrt[3]{n},$$

where $p_i(\alpha)$ - the probability of i-th transition, n - the length of a path.

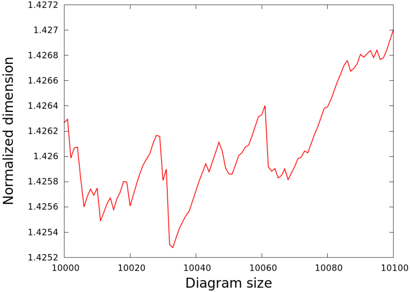
Normalized dimension

In 2D case the differences of normalized dimensions of typical and greedy paths are bounded.
In 3D case differences cannot be bounded if $C(\alpha) \neq C(\infty)$.
We have found that the asymptotics of 3D greedy and typical paths are the same if $\alpha = 0.558,\ C = 4/3$.
We expect that with this value of α the process becomes very close to a central one.

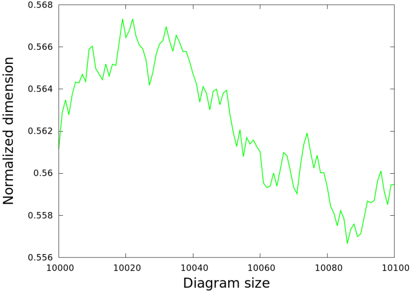
Normalized dimension



Normalized dimension



Greedy sequence

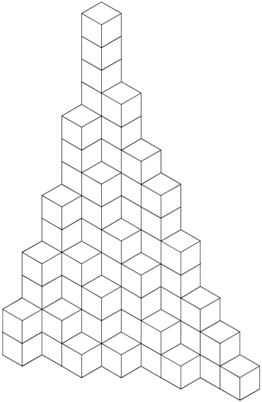


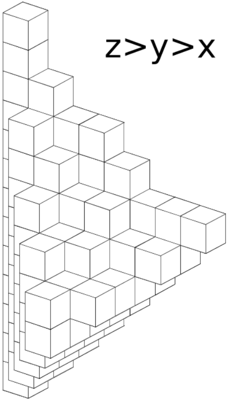
Typical radom sequence

Normalized dimension is a function with signifiant oscillations. The convergence to the upper limit is not proved, but it is confirmed by a large number of numerical experiments.

3D strict diagrams

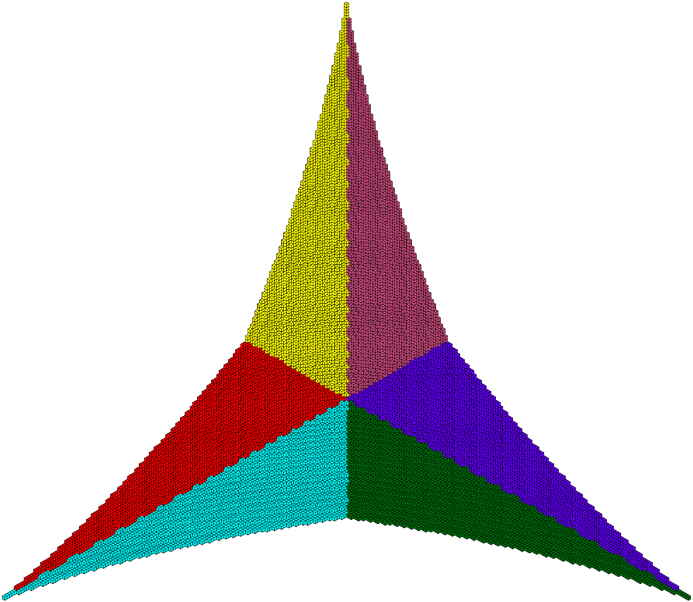
The geometric representation of strict diagrams:



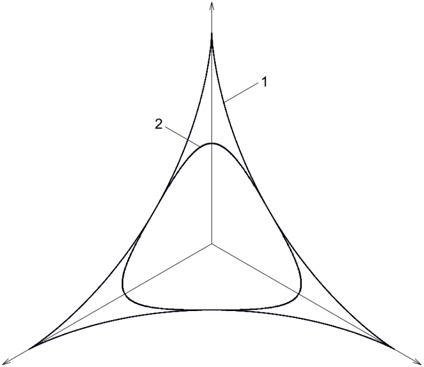


The correspondence between 3D standard and strict Young diagrams

The correspondence between strict diagrams of size n and standard symmetric diagrams of size $\approx 6n$. In addition to the point (x,y,z) we need to add all the other combinations: (x,z,y), (y,x,z), (y,z,x), (z,x,y), (z,y,x). The resulting standard diagram is obviously symmetric.



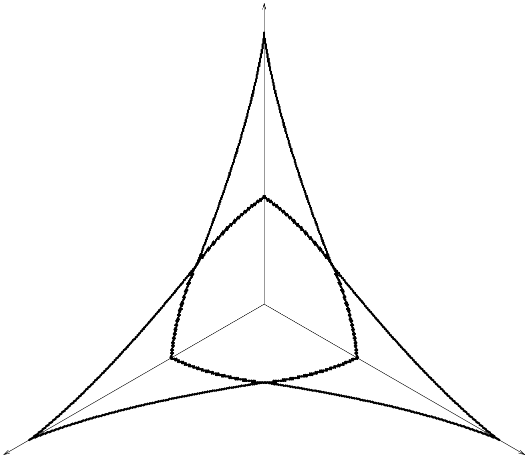
The limit shape of diagrams in 3D Pseudo-Plancherel process (Young graph)



1 - the contour of the front of the diagram of 15 million boxes from the greedy sequence,

2 - the contour of maximum closed section.

The limit shape of diagrams in 3D Pseudo-Plancherel process (Schur graph)



Previous works

- Vershik, A., and Pavlov, D. *"Numerical experiments in problems of asymptotic representation theory."*, 2010
- A. Okounkov, N. Reshetikhin, *"Correlation functions of Schur process with applications to local geometry of a random 3-dimensional Young diagram"*, 2013
- N. Vasilyev, V. Duzhin, *"Building Irreducible Representations of a Symmetric Group S(n) with Large and Maximum Dimensions"*, 2015
- N. N. Vasilyev and V. S. Duzhin, *"A study of the growth of maximal and typical normalized dimensions of strict Young diagrams"*, 2016
- V. S. Duzhin, N. N. Vasilyev, *"Asymptotic behavior of normalized dimensions of standard and strict Young diagrams — growth and oscillations"*, 2016
- N. N. Vasiliev, V. S. Duzhin, *"Numerical investigation of the asymptotics of the probabilities of paths in a Markov process on the 3D Young graph close to a central one"*, 2016

Conclusion

- An asymptotically central process was investigated on both 3D Young and Schur graphs;
- A normalized dimension was introduced for both cases;
- An asymptotic behaviour of normalized dimension was studied for standard and strict diagrams;
- The limit shapes of standard and strict diagrams were studied.

Future plans

- Investigation of the geometric properties of 3D strict Young diagrams;
- Investigation of asymptotic centrality of pseudo-Plancherel process on 3D Schur graph.

Thanks for your attention!