On the Stability Criteria for Hierarchical Three-Body Systems

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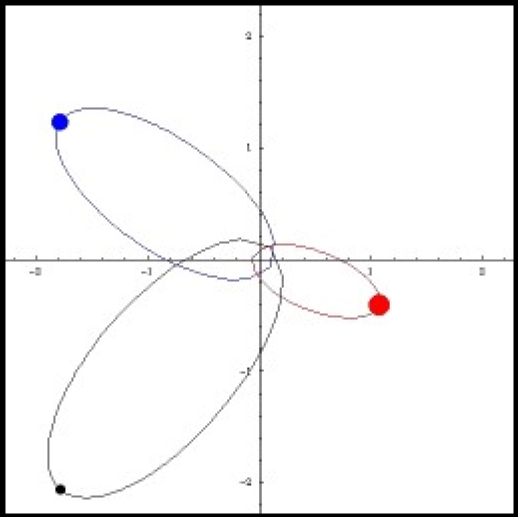
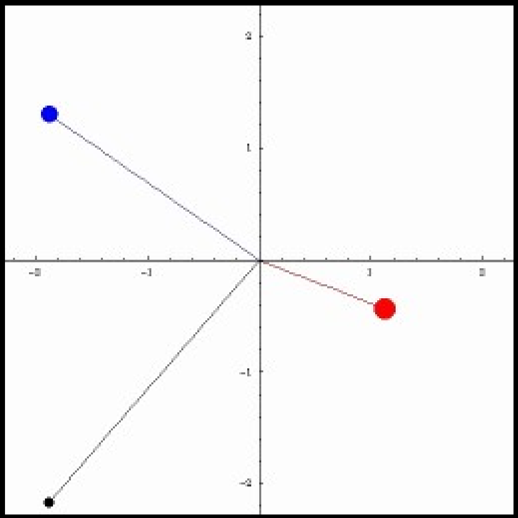
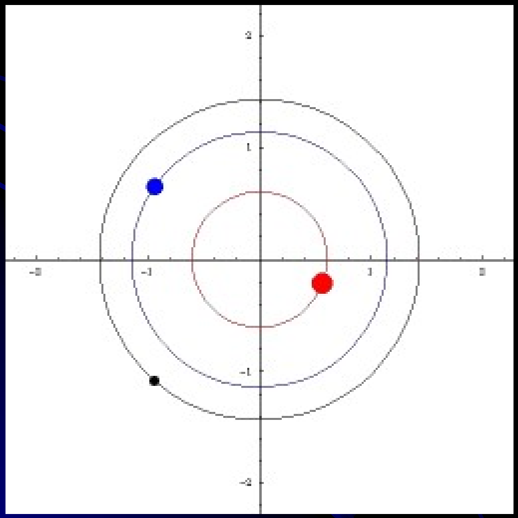
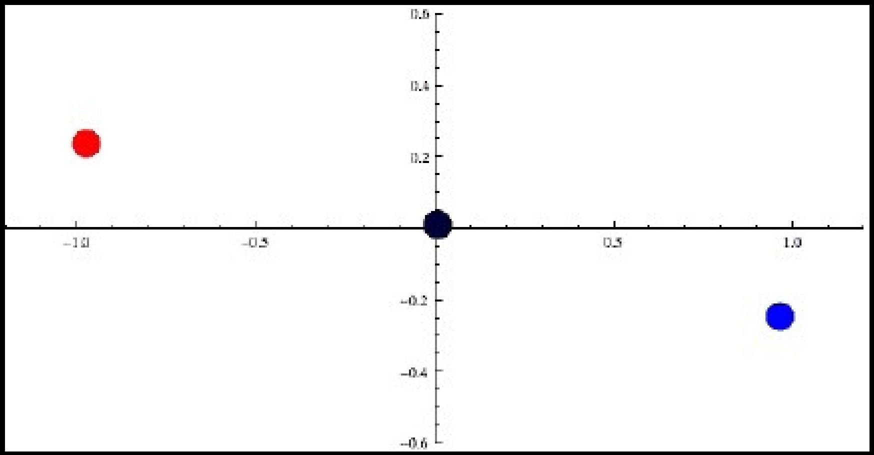
The three-body problem arises in many connections in stellar dynamics: three-body scattering drives the evolution of star clusters, and bound triple systems form long-lasting intermediate structures in them. Here we address the question of stability of triple stars. For a given system the stability is easy to determine by numerical orbit calculation. However, we often have only statistical knowledge of some of the parameters of the system. Then one needs a more general analytical formula. Here we tune coefficients for a theoretical stability limit formula in pericenter distance of the outer orbit for different mass combinations, outer orbit eccentricities and inclinations using results of numerical orbit calculations.

Three-body system



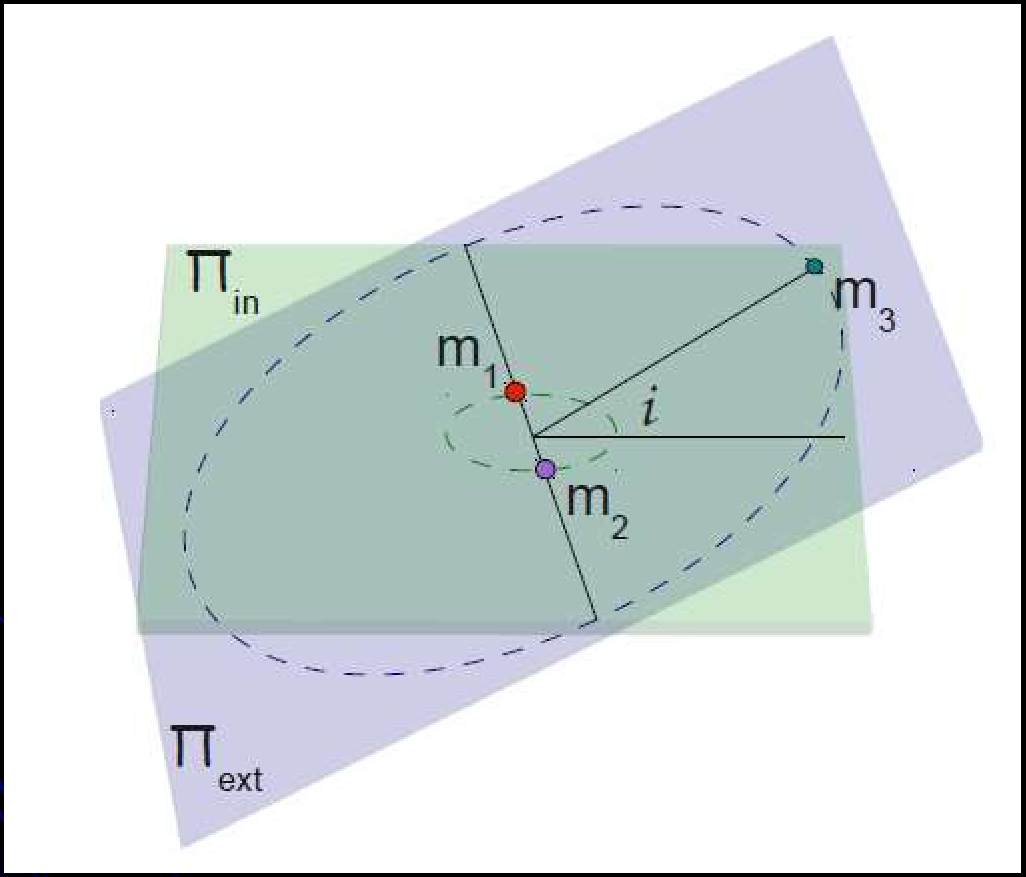






Animation by R. Moeckel

A hierarchical triple system consists of two bodies forming a binary system and a third body on a wider orbit.



Analytical stability criteria for hierarchical triple systems

1. V.G. Golubev (1967, 1968)

$$s = -\frac{c^2 H}{G^2 \left(\frac{M_1 + M_2 + M_3}{3}\right)^5},$$

where c and H are angular momentum and total energy of the triple system,

G is gravity constant,

$M_1 \geq M_2$ are the masses of inner binary components,

M_3 is the mass of the third body.

The critical value s_c depends on mass ratio and is found as a solution of the fifth power equation.

2. R.S. Harrington (1977)

$$F = \frac{a_{out}(1 - e_{out})}{a_{in}},$$

Critical value

$$F_c = A[1 + B\lg(\frac{1 + \frac{M_3}{M_1 + M_2}}{3/2})] + K.$$

Here A=3.50, B=0.70 for prograde motions;
A=2.75, B=0.64 for retrograde motions; K=2.
This is valid for spatial case when the orbits are far from orthogonal.

3. P.P. Eggleton and L.G. Kiseleva (1995)

$$X = \frac{P_{out}}{P_{in}},$$

$$X_c = \left(\frac{q_{out}}{1+q_{out}}\right)^{\frac{1}{2}} \left(\frac{1+e_{in}}{1-e_{out}}\right)^{\frac{3}{2}} Y_c^{\frac{3}{2}},$$

$$Y_c = 1 + \frac{3.7}{q_{out}^{\frac{1}{3}}} - \frac{2.2}{1+q_{out}^{\frac{1}{3}}} + \frac{1.4}{q_{in}^{\frac{1}{3}}} \cdot \frac{q_{out}^{\frac{1}{3}}-1}{q_{out}^{\frac{1}{3}}+1}.$$

Here $q_{in} = M_1/M_2$, $q_{out} = (M_1 + M_2)/M_3$. This is valid for spatial case when the orbits are far from orthogonal.

4. R. Mardling and S. Aarseth (1999)

$$Z = \frac{a_{out}(1 - e_{out})}{a_{in}(1 + e_{in})},$$

$$Z_c = 2.6\frac{(1 + e_{out})^{0.4}(1 + q)^{0.4}}{(1 - e_{out})^{0.0728}(1 + e_{in})^{1.2}}(1 - 0.3\frac{i}{\pi}).$$

Here $q = M_3/(M_1 + M_2)$,

i is the angle (in radians) between the vectors of angular momenta of inner and outer binaries.

5. R. Mardling and S. Aarseth (2003)

$$Z_2 = \frac{a_{out}(1 - e_{out})}{a_{in}},$$

$$Z_{c2} = 2.8 \cdot \left[(1+q) \cdot \frac{(1+e_{out})}{(1-e_{out})^{1/2}}\right]^{2/5}.$$

This is valid for spatial case when the orbits are far from orthogonal.

6. A. Tokovinin (2004)

$$C = \frac{P_{out}(1 - e_{out})^3}{P_{in}},$$

$$C_c = 5.0.$$

This is obtained from data for observed triple stars.

7. M. Valtonen and H. Karttunen (2006)

$$Q = \frac{a_{out}(1 - e_{out})}{a_{in}},$$

$$Q_c = 3.6 \left(1 + \frac{M_3}{M_1 + M_2}\right) \cdot (1 - e_{out})^{-1/11} \cdot \left(1 + e_{in}^2/2\right) \cdot$$

$$\cdot \left[0.07 + (1 + \cos\ i)^{1.15}\right]^{1/6}.$$

Idea of new criterion

The value of relative energy change 0.01 per orbit leads to $\Delta E/E=1$ after 10 000 orbits by random walk in energy, i.e. to instability. It appears that in this case the following expression for the inclination dependence should be used:

$$(1.75 + 0.5\cos i - \cos^2 i)^{1/3}$$

Idea of new criterion

The final formula for stability criterion for comparable masses (triple stars):

$$Q = \frac{a_{out}(1 - e_{out})}{a_{in}},$$

$$Q_c = 3(1+m_3/m_B)^{1/3}(1-e_{out})^{-1/6}(1.75+0.5\cos i-\cos^2 i)^{1/3}$$

Details see in

Valtonen, M.; Mylläri, A.; Orlov, V.; Rubinov, A. **The Problem of Three Stars: Stability Limit,** Dynamical Evolution of Dense Stellar Systems, Proceedings of the International Astronomical Union, IAU Symposium, Volume 246, p. 209-217, 2008

New criterion

$$Q_{st} = A\left(\lambda\sqrt{N}/(1-e_{out})\right)^{1/6}(f\cdot g)^{1/3}$$

$$\begin{aligned} f(e_{in},\cos i) &= \left\{1-\frac{2}{3}e_{in}\left[1-\frac{1}{2}e_{in}^2\right]-0.3\cos i\left[1\right.\right. \\ &- \left.\left.\frac{1}{2}e_{in}+2\cos i\left(1-\frac{5}{2}e_{in}^{3/2}-\cos i\right)\right]\right\} \end{aligned}$$

$$g(m_1,m_2,m_3) = \left(1+\frac{m_3}{m_1+m_2}\right)$$

$$A = 1.45 \pm 0.3$$

Testing of new criterion

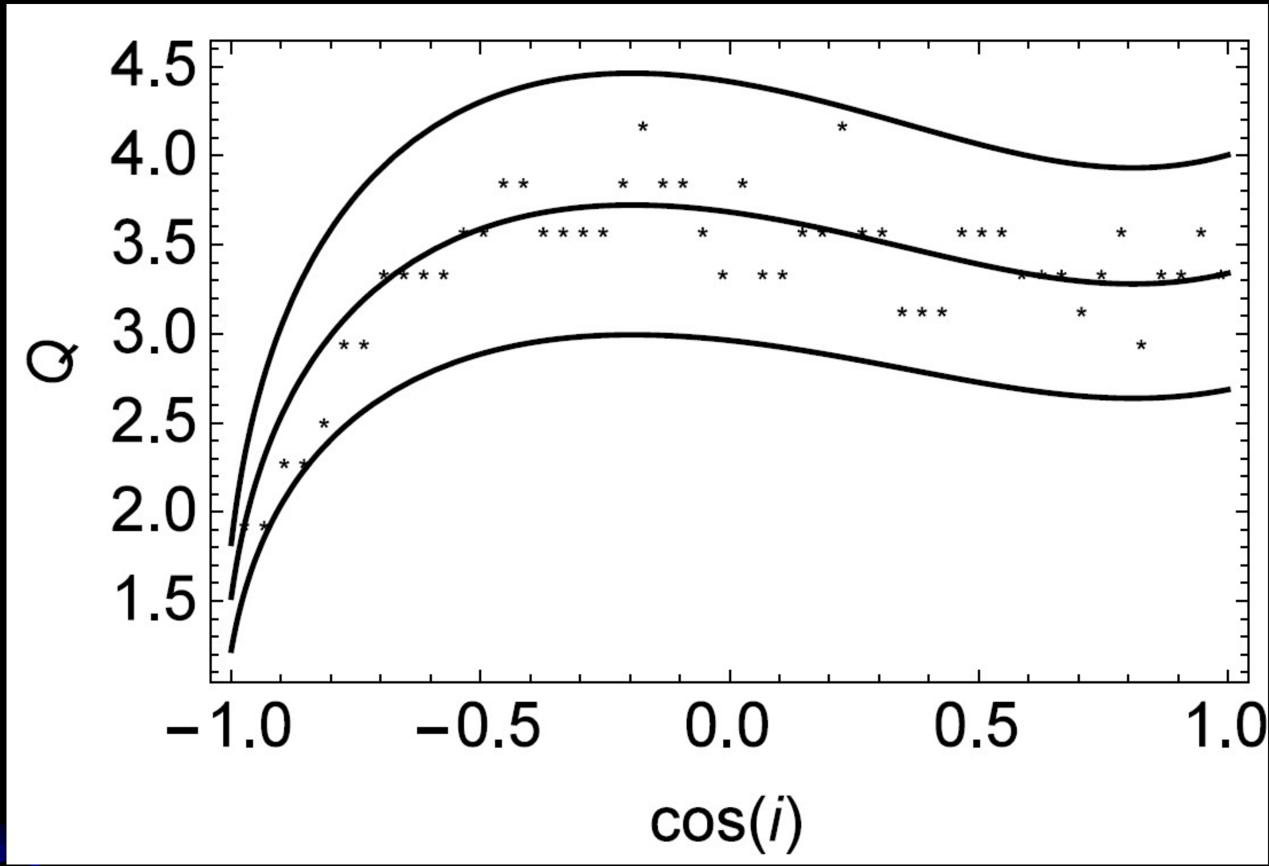


Figure 1. Stability limit Q calculations displayed with Q (y−axis) as a function of $\cos i$ (x−axis). The asterisc (*) shows the first unstable system when the Q-value is reduced in steps from high towards low values. Our standard parameter values are $m_1 = 1.5$, $m_2 = 0.5$, $m_3 = 0.5$, $e_{out} = 0.5$. In this case the inner eccentricity $e_{in} = 0.1$. The curves display the model outlined in the text. The lower curve is the always unstable limit (unstable below the line), the uppermost curve outlines the always stable region (higher up). The middle line is a cut of the middle of the mixed region and corresponds to $A = 1.45$. Upper and lower curves correspond to $A = 1.75$ and $A = 1.15$.

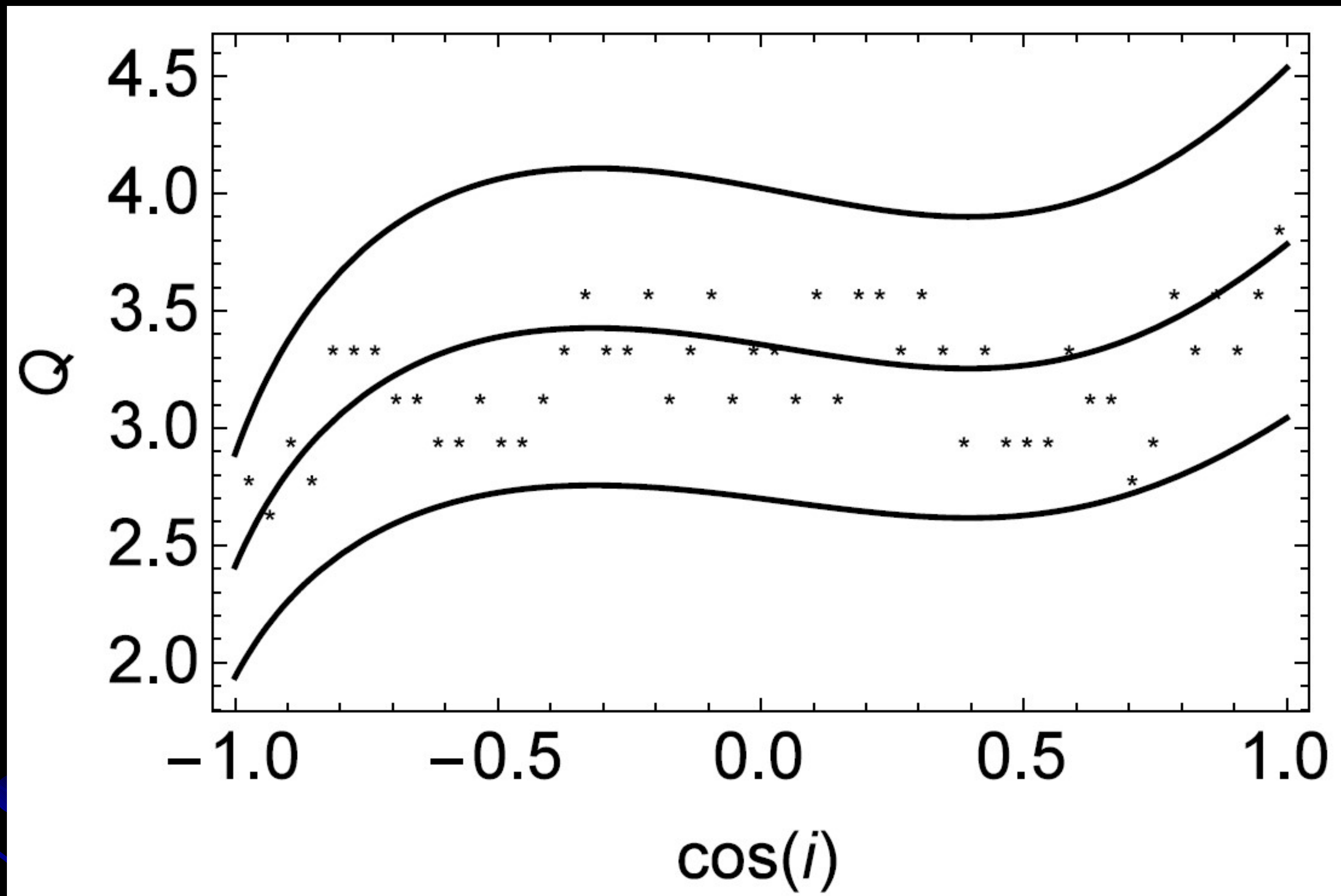
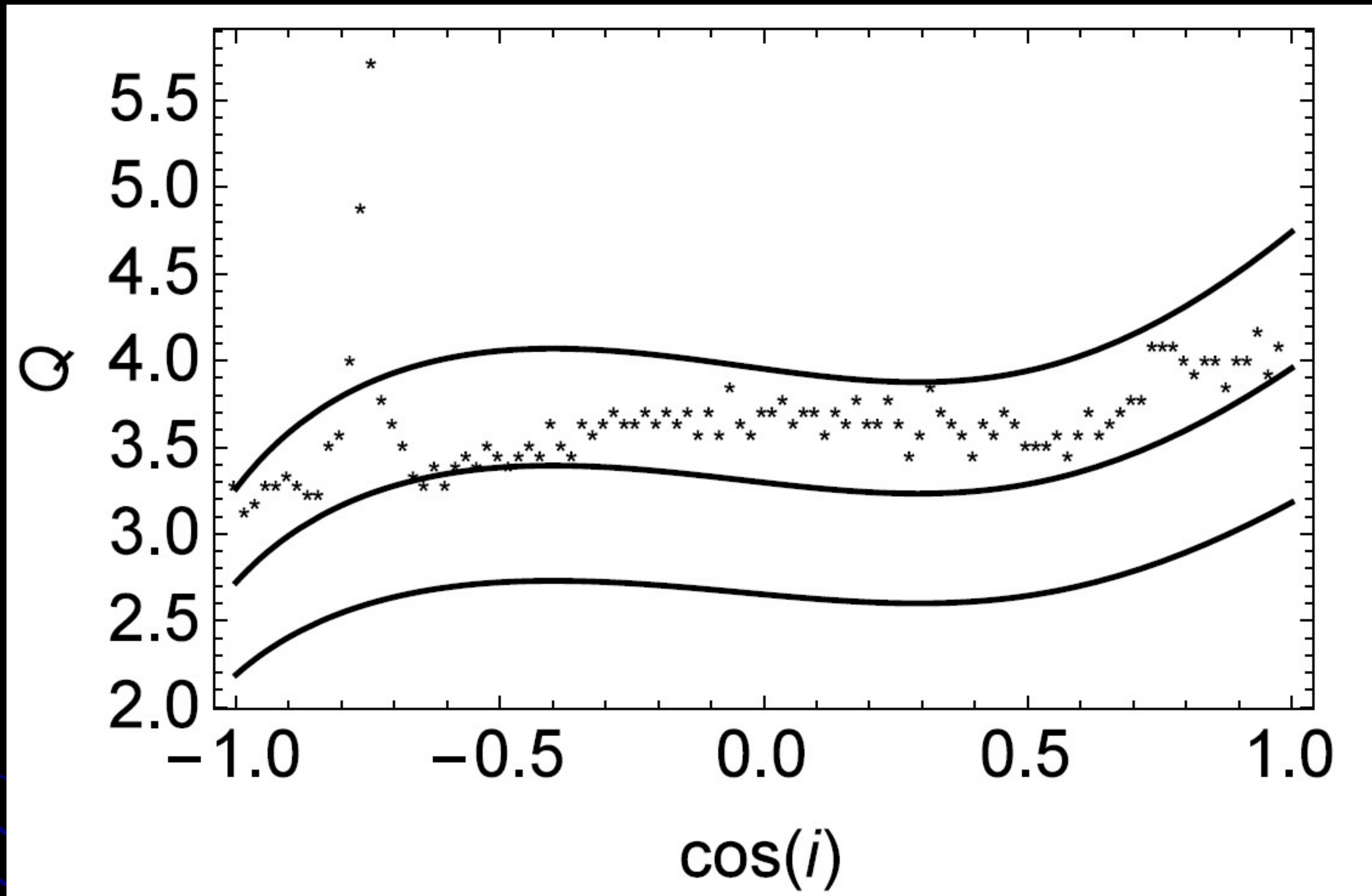
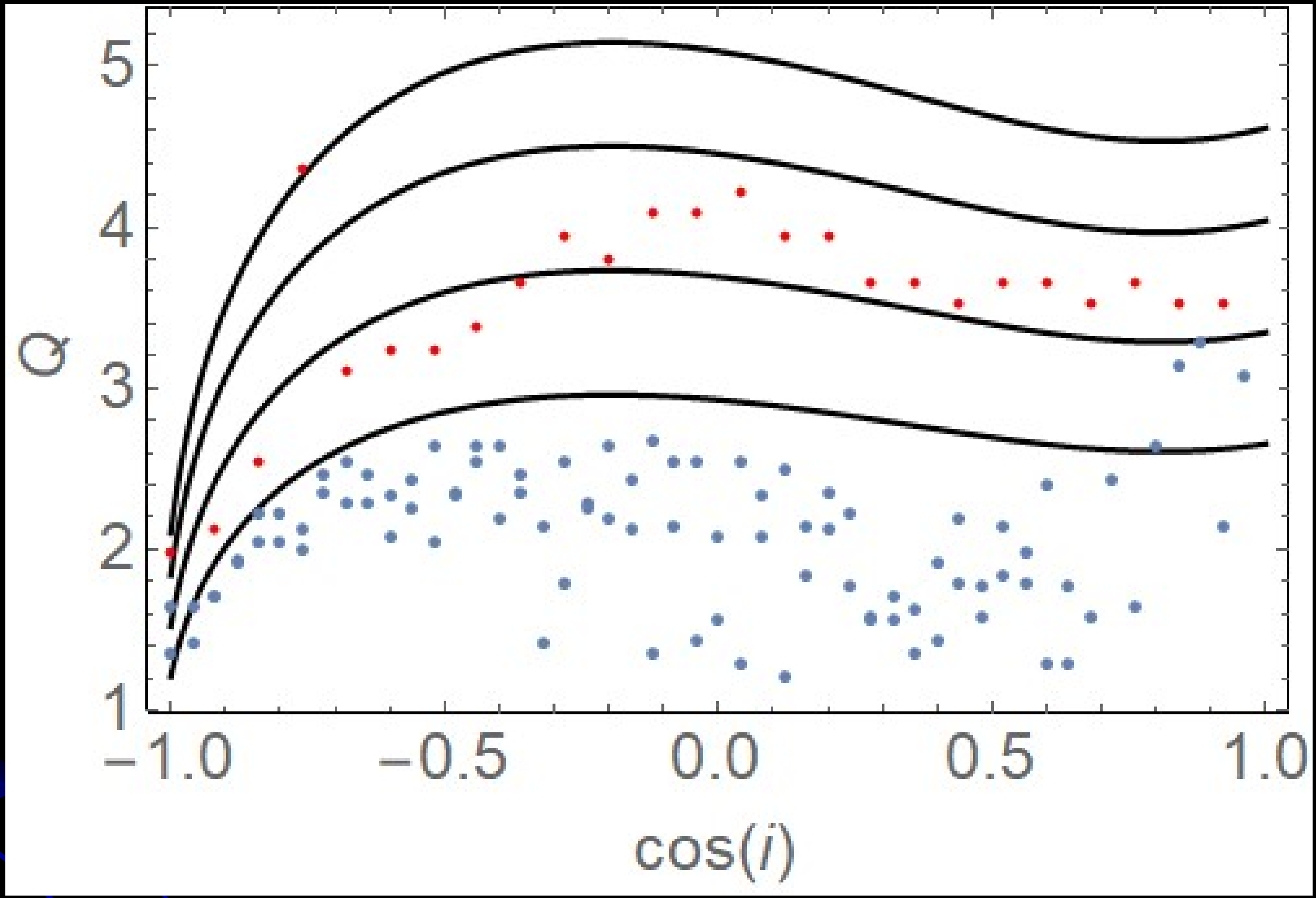


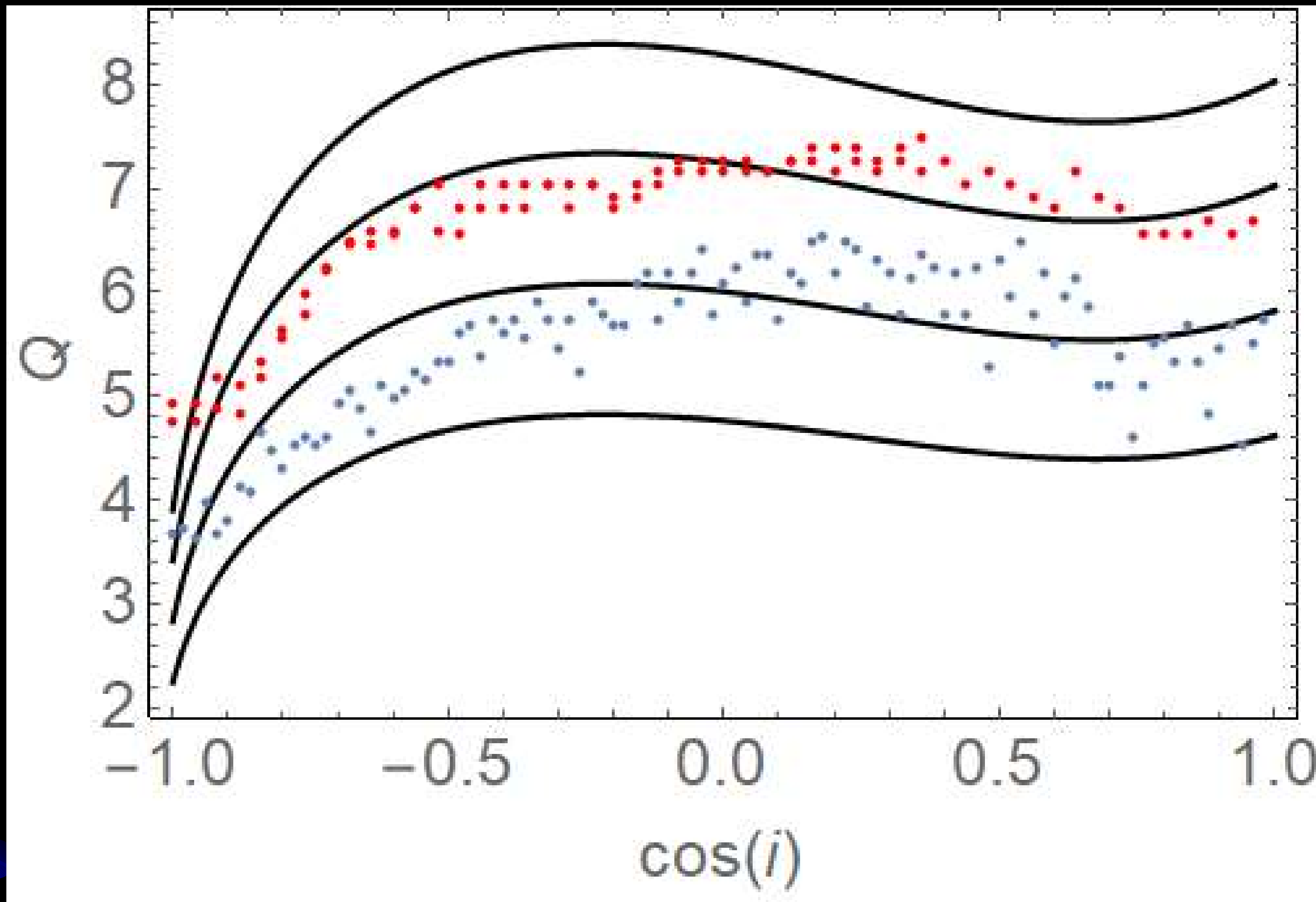
Figure 2. As above, but for inner eccentricity $e_{in} = 0.5$.



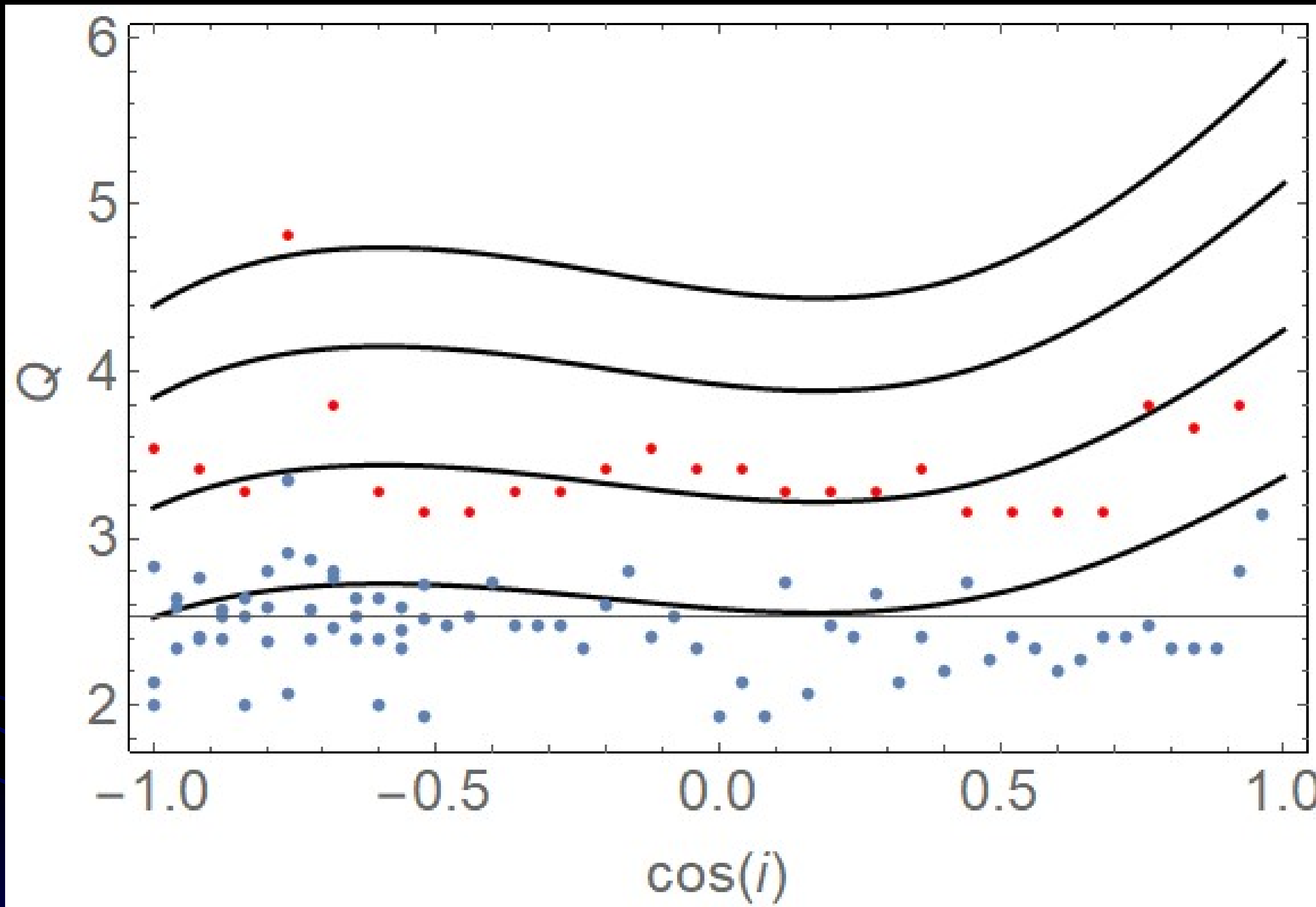
As above, but for the inner eccentricity $e_{in} = 0.6$



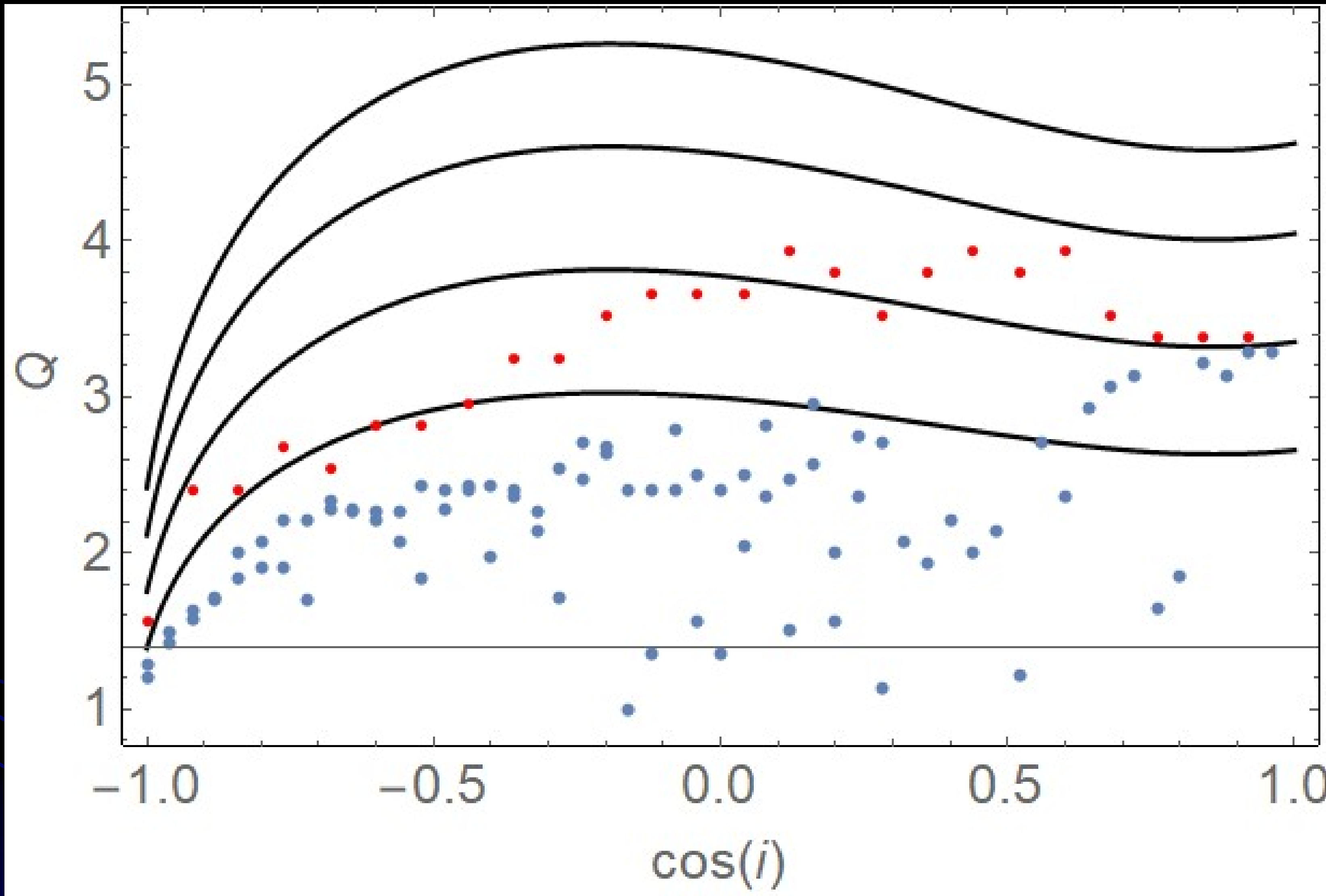
ein = 0.1; m1 = 1.5; m2=0.5; m3 = 0.5;
eout = 0.5;



ein = 0.25; m1 = 1.8; m2=0.2; m3 = 10.; eout = 0.5;

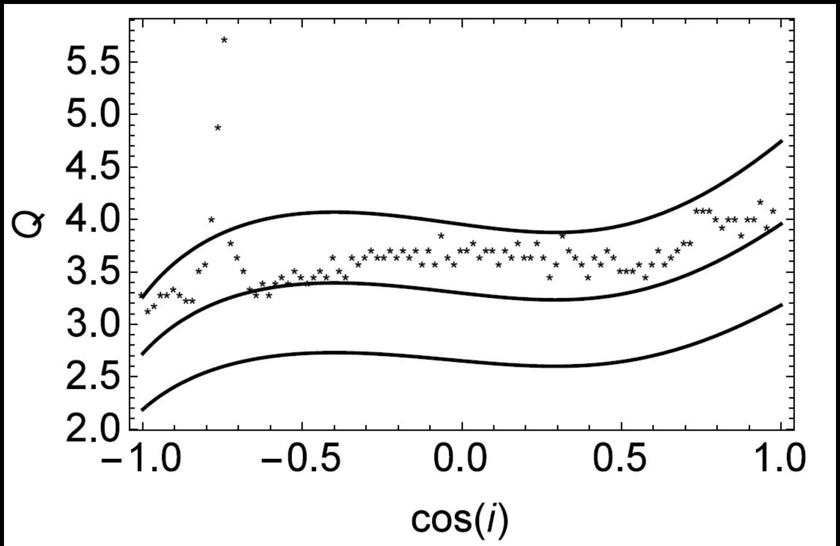


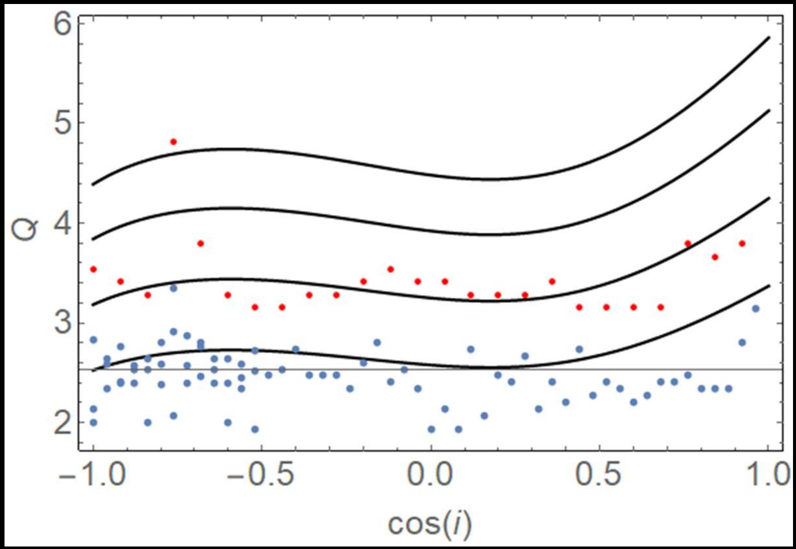
ein=0.75; m1=1.8; m2=0.2; m3=0.5; eout=0.5;

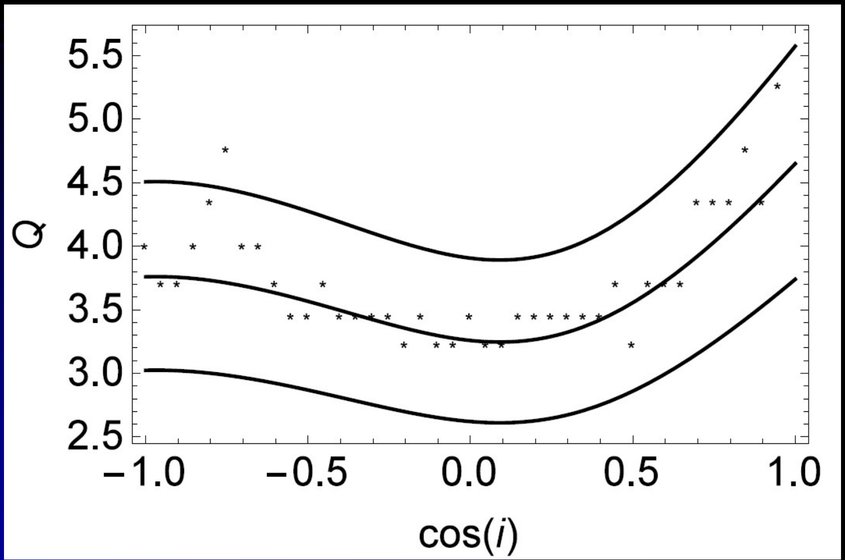


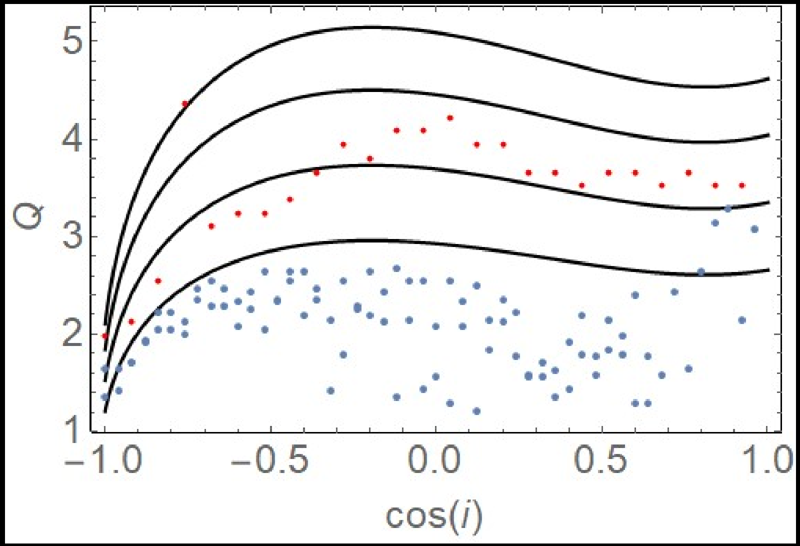
ein = 0.; m1 = 1.; m2=1., m3 = 0.5; eout = 0.5;

Resonance









Conclusions

1. The new stability criterion was suggested for hierarchical three-body systems. It is based on the theory of perturbations and random walking of the orbital elements of outer and inner binaries.

2. The numerical simulations have shown that a criterion is working very well in rather wide range of mass ratios (two orders at least).

3. New criterion shows a good correspondence with others and on comparison with the modeling it could be named among the best ones we tested.

Thank you for attention!

תודה על תשומת הלב!