The construction of averaged planetary motion theory by means CAS Piranha

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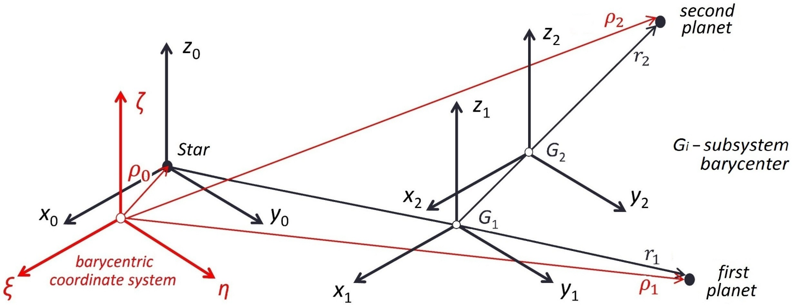
Subjects of this research

- we expand the Hamiltonian of four-planetary problem into the Poisson series in all orbital elements;
- we used the second system of Poincare elements and Jacobi coordinate system;
- the construction of the averaged Hamiltonian and motion equations in averaged elements is performed by Hori-Deprit method;
- these motion equations numerically integrated by Everhart method on a long time scales.

Jacobi coordinate system



From Barycentric frame to Jacobi frame

$$\mathbf{r}_0 = \frac{1}{\bar{m}_N}\boldsymbol{\rho}_0 + \mu \sum_{k=1}^{N} \frac{m_k}{\bar{m}_N}\boldsymbol{\rho}_k$$

$$\mathbf{r}_i = \boldsymbol{\rho}_i - \frac{1}{\bar{m}_{i-1}}\boldsymbol{\rho}_0 - \frac{\mu}{\bar{m}_{i-1}} \sum_{k=1}^{i-1} m_k \boldsymbol{\rho}_k$$

From Jacobi frame to Barycentric frame

$$\boldsymbol{\rho}_0 = \mathbf{r}_0 - \mu \sum_{k=1}^{N} \frac{m_k}{\bar{m}_N}\boldsymbol{r}_k$$

$$\boldsymbol{\rho}_i = \mathbf{r}_0 + \frac{\bar{m}_{i-1}}{\bar{m}_i}\mathbf{r}_i - \frac{\mu}{\bar{m}_{i-1}} \sum_{k=1}^{i-1} m_k \mathbf{r}_k$$

A star has mass m_0; planet k has mass $\mu m_0 m_k$, barycentric radius vector $\boldsymbol{\rho}_k$ and Jacobi radius vector $\mathbf{r}_k$; $\bar{m}_k = 1 + \mu m_1 + \ldots + \mu m_k$, μ is a small parameter.

The Hamiltonian of the problem

The Hamiltonian can be expressed in Jacobi coordinate system as shown here

$$h = -\sum_{i=1}^{N} \frac{M_i \kappa_i^2}{2a_i} + \mu \times G m_0 \Big\{ \sum_{i=2}^{N} \frac{m_i (2\mathbf{r}_i \mathbf{R}_i + \mu R_i^2)}{r_i \tilde{R}_i (r_i + \tilde{R}_i)} - \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{m_i m_j}{|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|} \Big\}.$$

This is a difference of radius vectors in barycentric frame

$$|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j| = \mathbf{r}_i - \mathbf{r}_j + \mu \sum_{k=j}^{i-1} \frac{m_k}{\bar{m}_k} \mathbf{r}_k.$$

Quantities $\mathbf{R}_i$ and $\tilde{R}_i$ are defined as

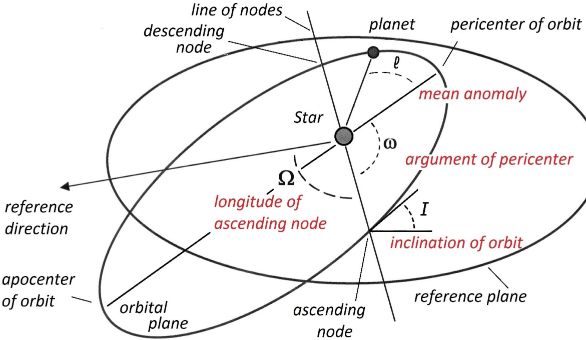
$$\mathbf{R}_i = \sum_{k=1}^{i} \frac{m_k}{\bar{m}_k} \mathbf{r}_k, \quad \tilde{R}_i = \sqrt{r_i^2 + 2\mu \mathbf{r}_i \mathbf{R}_i + \mu^2 R_i^2}.$$

Here $1 \leq j < i \leq N$; G – gravitational constant, $M_i = m_i \bar{m}_{i-1} / \bar{m}_i$ – reduced mass, $\kappa_i^2 = G m_0 \bar{m}_i / \bar{m}_{i-1}$ – gravitational parameter, a_i – semi-major axis.

Small parameter μ is the ratio of sum of planet masses to the mass of the star.

For Solar system the value of μ can take equal to 0.001. This is expansion parameter.

The second system of Poincare elements



The second system of Poincare elements allows sufficiently simplifying an angular part of the series.

These elements are defined through classical Kepler orbital elements.

$$L = M\sqrt{\kappa^2 a} \qquad \lambda = \omega + \Omega + l$$

$$\xi_1 = \sqrt{2L(1-\sqrt{1-e^2})}\cos(\omega+\Omega) \qquad \eta_1 = -\sqrt{2L(1-\sqrt{1-e^2})}\sin(\omega+\Omega)$$

$$\xi_2 = \sqrt{2L\sqrt{1-e^2}(1-\cos I)}\cos\Omega \qquad \eta_2 = -\sqrt{2L\sqrt{1-e^2}(1-\cos I)}\sin\Omega$$

Kepler orbital elements: a – semi-major axis, e – eccentricity, I – inclination, ω – argument of pericenter, Ω – longitude of ascending node, l – mean anomaly.

The expansion of the Hamiltonian into the Poisson series

We can write the expansion of the planetary system Hamiltonian as shown here

$$h = h_0 + \mu h_1 = h_0 + \sum_{k,p,n} A_{kpn} \mu^k x^p \cos n\lambda,$$

where h_0 – undisturbed Hamiltonian, h_1 – the disturbing function of the system, A_{kpn} – series coefficients, x^p is product of Poincare elements with corresponding degrees, $n\lambda$ – is linear combination of mean longitudes.

$$x^p = L_1^{p1} \xi_{1,1}^{p2} \eta_{1,2}^{p3} \xi_{2,1}^{p4} \eta_{2,2}^{p5} \cdots L_4^{p16} \xi_{1,4}^{p17} \eta_{1,4}^{p18} \xi_{2,4}^{p19} \eta_{2,4}^{p20}$$

$$n\lambda = n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 + n_4\lambda_4$$

All orbital elements and mass parameters are saved in series expansion as symbol variables. Series coefficients are rational numbers (integer numerators and denominators) with arbitrary precision.

For our calculations we used a personal computer with six-core processor Intel Core-i7 with a frequency of 3300 MHz and available RAM of 128 Gb.

Computer algebra system Piranha

We used computer algebra system (CAS) Piranha for construction of the Hamiltonian expansion and implementation of the algorithm of Hori-Deprit method. Piranha is an echeloned Poisson series processor.

The author of this program is Dr. Francesco Biscani (Interdisciplinary Center for Scientific Computing, Heidelberg, Germany).

Piranha is multi-platform C++ program with Python's interface for analytical manipulations with different series types.

Piranha can works with

- polynomial series
- Fourier series
- Poisson series
- echeloned Poisson series (Poisson series with denominators)

Types of coefficients and polynomial degrees

- integer
- real
- rational

Piranha's source code is available on www.github.com/bluescarni/piranha.

Piranha's features

- summation and multiplication of series by using operators $+$, $-$ and $*$
- computation the multiplicative inverse of series *arg* by *invert(arg)*
- construction binomial expansions like as $(1+x)^r$, where x is a series, $r \in \mathbb{Q}$
- global series truncation by *set_auto_truncate_degree(max_degree, names_list)*
- cutting off unnecessary terms by *truncate_degree(arg, max_degree, names_list)*
- substitution series/value *x* into variable *name* of series *arg* by *subs(arg, name, x)*
- evaluation series arg by set of numerical values by using *evaluate(arg, eval_dict)*
- partial derivative w.r.t. variable *name* giving by *partial(arg, name)*
- integral over variable *name* giving by *integrate(arg, name)*
- time integral of trigonometric part of series arg is *arg.t_integrate(name)*
- *cos(arg)*, *sin(arg)*, *pbracket(f, g, p_list, q_list)*
- *transformation_is_canonical(new_p, new_q, p_list, q_list)*

The algorithm of the expansion. Base series

We implemented simple Poincare processor for construction of expansions

- Cartesian coordinates x, y, z
- distance r and inverse distance $1/r$

We expressed these quantities through Bessel functions. In instance, r, $1/r$ look as

$$\frac{r}{a} = 1 + \frac{e^2}{2} + \sum_{k=1}^{\infty} \frac{e}{k} \left(J_{k+1}(ke) - J_{k-1}(ke)\right) \cos kl,$$

$$\frac{a}{r} = 1 + 2\sum_{k=1}^{\infty} J_k(ke) \cos kl$$

$J_k(ke)$ – Bessel function,
e – eccentricity,
l – mean anomaly,
a – semi-major axis

Then we written e and $\cos l$, $\sin l$ through Poincare elements. After that, we obtained:

- ratio r_i/r_j from $1/r_j$ and r_i
- scalar product $\mathbf{r}_i\mathbf{r}_j$ from x, y, z

The inverse distance $1/\Delta_{ij} = |\mathbf{r}_i - \mathbf{r}_j|^{-1}$ is expanded into series as

$$\frac{1}{\Delta_{ij}} = \frac{1}{r_j}\left(1 + \left(\frac{r_i}{r_j}\right)^2 - 2\left(\frac{r_i}{r_j}\right)\cos H\right)^{-\frac{1}{2}} = \frac{1}{r_j}\sum_{n=0}^{\infty}\left(\frac{r_i}{r_j}\right)^n P_n(\cos\theta_{ij}),$$

P_n – Legendre polynomial of degree n, θ_{ij} – an angle between vectors $\mathbf{r}_i$ and $\mathbf{r}_j$.

Finally Legendre polynomials are expressed through Poincare elements.

The algorithm of the expansion. The disturbing function

Terms of the main part of the disturbing function can be expressed as

$$\frac{1}{|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|} = \frac{1}{\Delta_{ij}}\left(1 - \frac{2\mu A_{ij} + \mu^2 B_{ij}}{\Delta_{ij}^2}\right)^{-\frac{1}{2}} = \frac{1}{\Delta_{ij}} - \mu\frac{A_{ij}}{\Delta_{ij}^3} + \mu^2\left(\frac{3}{2}\frac{A_{ij}^2}{\Delta_{ij}^5} - \frac{1}{2}\frac{B_{ij}}{\Delta_{ij}^3}\right) + \ldots$$

Quantities A_{ij}, B_{ij} are defined as

$$A_{ij} = (\mathbf{r}_i - \mathbf{r}_j)\sum_{k=j}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k, \; B_{ij} = \left(\sum_{k=j}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k\right)^2.$$

Terms of the second part of the disturbing function can be expressed as

$$\frac{2\mathbf{r}_i\mathbf{R}_i + \mu\mathbf{R}_i^2}{r_i\tilde{R}_i(r_i + \tilde{R}_i)} = \frac{C_i}{r_i^3} + \mu\left(-\frac{3}{2}\frac{C_i^2}{r_i^5} + \frac{1}{2}\frac{D_i}{r_i^3}\right) + \mu^2\left(\frac{5}{2}\frac{C_i^3}{r_i^7} - \frac{3}{2}\frac{C_iD_i}{r_i^5}\right) + \ldots$$

Quantities C_i, D_i are defined as

$$C_i = \mathbf{r}_i\sum_{k=1}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k, \; D_i = B_{i1} = \left(\sum_{k=1}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k\right)^2.$$

Properties of constructed series

Table 1 : Base series

n	series	$x,\ y$	z	r	$1/r$	r_i/r_j	$\cos\theta_{ij}$	$1/\Delta_{ij}$
5	number	96	116	46	41	3 486	2 438	591 376
	accuracy	10^{-9}	10^{-7}	10^{-9}	10^{-8}	10^{-8}	10^{-7}	10^{-5}
	time	0.05^s	0.05^s	0.15^s	0.15^s	0.05^s	0.30^s	96^s

Table 2 : Expansion of the Hamiltonian

items	n_1, p_1	N_1	Δs_1	t	n_2, p_2	N_2	Δs_2	t
i, j	Main part of the disturbing function							
$1, 2$	5, 25	591 376	10^{-8}	96^s	3, 15	175 786	10^{-5}	42^s
$2, 3$	5, 25	591 376	10^{-8}	96^s	3, 15	175 786	10^{-5}	42^s
$3, 4$	5, 25	591 376	10^{-7}	96^s	3, 15	175 786	10^{-4}	42^s
$1, 3$	5, 20	385 460	10^{-8}	84^s	2, 10	82 874	10^{-4}	13^s
$2, 4$	5, 20	385 460	10^{-10}	84^s	2, 10	82 874	10^{-4}	13^s
$1, 4$	5, 15	223 476	10^{-11}	71^s	2, 5	41 988	10^{-5}	10^s
k	Second part of the disturbing function							
$2, 3, 4$	5, –	12 852	10^{-8}	6^s	3, –	7 994	10^{-5}	3^s
Hamiltonian expansion		$N_1 = 2\ 768\ 524$			$N_2 = 735\ 094$		$\Delta s = 10^{-12}$	

For terms with μ^k: n_k – max. degree of elements, p_k – max. order of P_n, N_k – number of terms, Δs_k – relative accuracy of the expansion (for Solar system's giant planets); t – time.

Hori-Deprit (or Lie transformation) method

The averaged Hamiltonian can be expressed as

$$H(X) = H_0(X) + \sum_{m=1}^{\infty} \mu^m H_m(X).$$

We can find quantities H_m using by the general equation of Hori-Deprit method

$$H_m = h_m + \sum \frac{1}{r!}\{T_{j_r}, \{\dots, \{T_{j_1}, h_{j_0}\}\}\},$$

where $0 \le j_0 \le m-1$; $j_1, j_2, \dots, j_r \ge 1$; $\sum_{s=0}^{k} j_s = m$; $1 \le r \le m$. h_m – items of non-averaged Hamiltonian, T_{j_r} – items of the generationg function (defined below).

Figure brackets are Poisson brackets, which are defined as

$$\{f, g\} = \sum_k \Big(\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k}\Big).$$

f, g – some functions of Poincare elements, $p = \{L_1, \cdots, L_4, \xi_{1,1}, \xi_{1,2} \cdots \eta_{1,4}\}$ – vector of momentum, $q = \{\lambda_1, \cdots, \lambda_4, \xi_{2,1}, \xi_{2,2} \cdots \eta_{2,4}\}$ – vector of coordinates.

Hori-Deprit method

On each step we can write that

$$H_m = \{T_m, h_0\} + \Phi_m \quad \text{or} \quad \Phi_m = H_m + \sum_{k=1}^{N} \omega_k \frac{\partial T_m}{\partial \lambda_k},$$

where Φ_m is defined on previously step, $\omega_k = \partial h_0 / \partial L_k$ – frequency of motion.

After time integration of $(\Phi_m - H_m)$ we get a solution for T_m

$$T_m(x, \lambda) = \sum \frac{B_{pn}}{n\omega} x^p \sin n\lambda,$$

where B_{pn} – coefficient of echeloned Poisson series, $n\omega$ – linear combination of motion frequences.

We can explicitly write that (up to terms with μ^2)

$$H_0 = h_0,$$
$$H_1 = \{T_1, h_0\} + h_1,$$
$$H_2 = \{T_2, h_0\} + \{T_1, h_1\} + \frac{1}{2}\{T_1, \{T_1, h_0\}\}.$$

The motion equations

The motion equations in averaged elements X, Λ are shown here

$$\boxed{\frac{\mathrm{d}X}{\mathrm{d}t} = \{H, X\}, \qquad \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \{H, \Lambda\}.}$$

The transformation between averaged X, Λ and osculating x, λ elements are given by the functions for the change of variables. They are constructed by using the general equation of Hori-Deprit method

$$u_{im} = \sum \frac{1}{r!}\{T_{j_r}, \{\ldots, \{T_{j_1}, x\}\}\}, \quad i = \overline{1, 20}$$

$$v_{im} = \sum \frac{1}{r!}\{T_{j_r}, \{\ldots, \{T_{j_1}, \lambda\}\}\}, \quad i = \overline{1, 4}.$$

Here the domain of summation is $j_1, j_2, \ldots, j_r \geq 1$; $\sum_{s=0}^{k} j_s = m$; $1 \leq r \leq m$.

After that we can take averaged elements as

$$X = x + \sum_{m=1}^{\infty} (-1)^m \mu^m u_m(x, \lambda),$$

$$\Lambda = \lambda + \sum_{m=1}^{\infty} (-1)^m \mu^m v_m(x, \lambda).$$

Properties of the motion equations

Table 3 : Properties of the averaged Hamiltonian and functions Φ_1, Φ_2

	H_0	Φ_1	H_1	Φ_2	H_2
number	4	2 781 376	6 393	2 927 013 173	381 534
time	0	taken together: $12^{\mathrm{m}}30^{\mathrm{s}}$		taken together: $66^{\mathrm{h}}40^{\mathrm{m}}$	

Table 4 : Properties of the averaged motion equations

el.	N_1	t	N_2	t	el.	N_1	t	N_2	t
L_1	$\dot{L}_k = 0\ \forall k$, i.e.				λ_1	2 936	0.2^{s}	302 476	47^{s}
L_2	in averaged theory				λ_2	3 452	0.2^{s}	361 534	57^{s}
L_3	semi-major axes				λ_3	3 453	0.2^{s}	347 284	55^{s}
L_4	are constant				λ_4	2 939	0.2^{s}	255 418	40^{s}
$\xi_{1,1}$	906	0.05^{s}	33 400	5.3^{s}	$\xi_{2,1}$	911	0.05^{s}	21 638	3.4^{s}
$\xi_{1,2}$	1 063	0.05^{s}	42 322	6.7^{s}	$\xi_{2,2}$	1 071	0.05^{s}	27 679	4.4^{s}
$\xi_{1,3}$	1 060	0.05^{s}	42 576	6.8^{s}	$\xi_{2,3}$	1 071	0.05^{s}	27 971	4.5^{s}
$\xi_{1,4}$	897	0.05^{s}	33 754	5.4^{s}	$\xi_{2,4}$	911	0.05^{s}	22 316	3.5^{s}

Notes:

- data for elements η_1, η_2 are the same for ξ_1, ξ_2;
- calculation time t consists time of series loading and time for series saving.

Properties of the functions for the change

Table 5 : Properties of the generating function

	T_1	T_2
number	2 774 983	2 926 631 639
time	$3^{\mathrm{m}}50^{\mathrm{s}}$	$19^{\mathrm{h}}40^{\mathrm{m}}$

Table 6 : Properties of the functions for the change of variables

el.	N_1	t	N_2	t	el.	N_1	t	N_2	t
L_1	$1.16{\cdot}10^6$	64^{s}	$1.86{\cdot}10^9$	28.5^{h}	λ_1	$2.36{\cdot}10^6$	131^{s}	Not calculated;	
L_2	$1.52{\cdot}10^6$	84^{s}	$2.29{\cdot}10^9$	35.2^{h}	λ_2	$3.09{\cdot}10^6$	172^{s}	values of function	
L_3	$1.52{\cdot}10^6$	84^{s}	$2.30{\cdot}10^9$	35.3^{h}	λ_3	$3.09{\cdot}10^6$	172^{s}	of change are less	
L_4	$1.16{\cdot}10^6$	64^{s}	$1.88{\cdot}10^9$	28.8^{h}	λ_4	$2.36{\cdot}10^6$	131^{s}	than in 1st appox.	
$\xi_{1,1}$	$0.49{\cdot}10^6$	27^{s}	$309 \cdot 10^6$	4.7^{h}	$\xi_{2,1}$	$0.45{\cdot}10^6$	25^{s}	$197 \cdot 10^6$	3.0^{h}
$\xi_{1,2}$	$0.64{\cdot}10^6$	36^{s}	$401 \cdot 10^6$	6.3^{h}	$\xi_{2,2}$	$0.58{\cdot}10^6$	32^{s}	$258 \cdot 10^6$	4.0^{h}
$\xi_{1,3}$	$0.64{\cdot}10^6$	36^{s}	$398 \cdot 10^6$	6.2^{h}	$\xi_{2,3}$	$0.58{\cdot}10^6$	32^{s}	$256 \cdot 10^6$	3.9^{h}
$\xi_{1,4}$	$0.49{\cdot}10^6$	27^{s}	$305 \cdot 10^6$	4.7^{h}	$\xi_{2,4}$	$0.45{\cdot}10^6$	25^{s}	$194 \cdot 10^6$	3.0^{h}

Notes:

- data for elements η_1, η_2 are the same for ξ_1, ξ_2;
- calculation time t consists time of series loading and time for series saving.

Initial conditions for integration of the motion equations

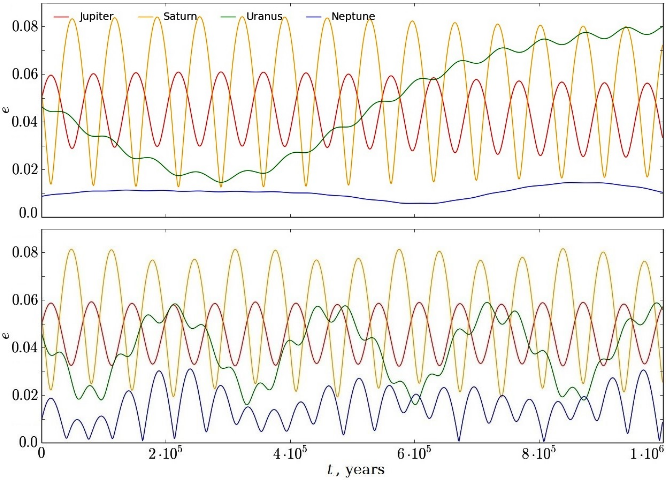
- Integration of the motion equations in averaged Poincare elements is performed by Everhart method of 15th order for both approximations of the motion theory.
- Averaged Poincare elements for initial conditions are obtained using the functions for the the change of variables in second approximation.
- Initial conditions for integration are given in the table below.

Table 7 : Averaged orbital elements of Solar system's giant planets in Jacobi coordinates, w.r.t. mean ecliptic and equinox J2000.0 (on date 01.01.2000)

planet	a, AU	e	I, °	ω, °	Ω, °	l, °
Jupiter	5.203433	0.0487429	1.303256	273.9763	100.4572	19.8694
Saturn	9.554264	0.0551739	2.488183	339.6182	113.7219	316.7210
Uranus	19.216003	0.0465882	0.773159	97.9370	73.9733	142.0651
Neptune	30.105333	0.0086852	1.775393	273.6950	131.7606	258.9275

planet	L	ξ_1	η_1	ξ_2	η_2	λ, °
Jupiter	0.037446	$9.1223 \cdot 10^{-3}$	$-2.5471 \cdot 10^{-3}$	$-8.0161 \cdot 10^{-4}$	$-4.3301 \cdot 10^{-3}$	34.3029
Saturn	0.015195	$-2.3241 \cdot 10^{-4}$	$-6.5801 \cdot 10^{-3}$	$-2.1466 \cdot 10^{-3}$	$-4.8901 \cdot 10^{-3}$	50.0611
Uranus	0.003292	$-2.6194 \cdot 10^{-3}$	$-3.6558 \cdot 10^{-4}$	$2.1329 \cdot 10^{-4}$	$-7.4266 \cdot 10^{-4}$	313.9754
Neptune	0.004801	$4.6506 \cdot 10^{-4}$	$-3.6343 \cdot 10^{-4}$	$-1.4309 \cdot 10^{-3}$	$-1.6117 \cdot 10^{-3}$	304.3832

Results of integration. Eccentricities



pl.	period	ampl.
J	68551	0.018
S	68551	0.036
U	$1.38{\cdot}10^6$	0.035
N	$\sim 5{\cdot}10^5$	0.005

pl.	period	ampl.
J	65910	0.014
S	65910	0.032
U	259619	0.023
N	42628	0.016

Figure 1 : Evolution of eccentricities for the theory of the first order (up) and the second order (down) on time interval 10^6 yr.

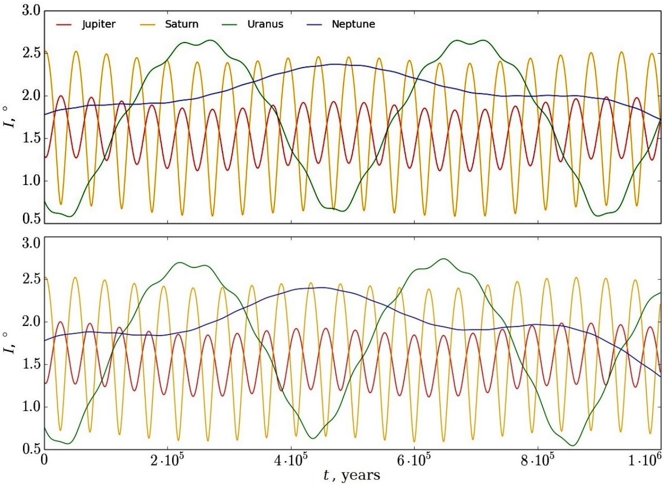
Results of integration. Small denominators

Qualitative differences between evolution of Uranus and Neptune's eccentricities in both orders of the motion theory explained the presence of small denominators in the motion equations of the second order.
Denominators (linear combinations of motion frequencies) which give maximum contribution to the motion equations of Jupiter (J), Saturn (S), Uranus (U) and Neptune (N) show in the table below.

planets	frequency	value	planets	frequency	value
J	$(\nu_1 - 2\nu_2)$	2.848^{-4}	S, U	$(\nu_2 - 3\nu_3)^2$	9.0536^{-10}
J, S	$(\nu_1 - 2\nu_2)^2$	8.112^{-8}	S	$(\nu_2 - 3\nu_3)^3$	-2.724^{-14}
J, S	$(\nu_1 - 2\nu_2)^3$	2.310^{-11}	U, N	$(\nu_3 - 2\nu_4)$	-4.066^{-6}
J	$(\nu_1 - 3\nu_2)$	-1.978^{-4}	U, N	$(\nu_3 - 2\nu_4)^2$	1.653^{-11}
J, S	$(\nu_1 - 3\nu_2)^2$	8.866^{-8}	U, N	$(\nu_3 - 2\nu_4)^3$	-6.720^{-17}

We can try to exclude the influence of small denominators by transfer them to the third approximation of the motion theory.

Results of integration. Inclinations

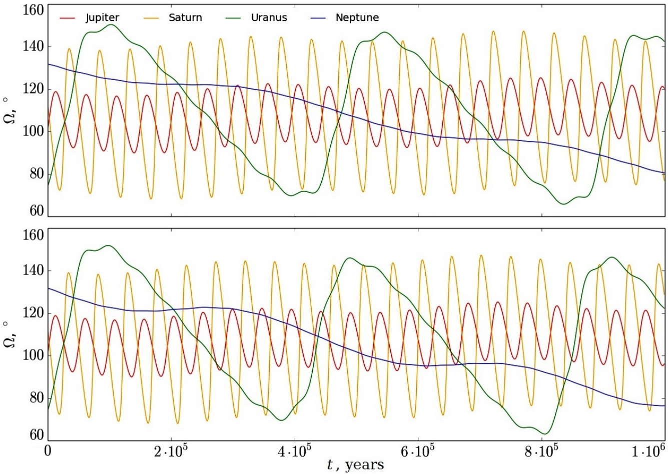


pl.	period	ampl.
J	49154	0.486
S	49154	1.017
U	433220	1.133
N	$1.89{\cdot}10^6$	0.803

pl.	period	ampl.
J	47822	0.483
S	47822	1.014
U	405482	1.192
N	$1.66{\cdot}10^6$	0.829

Figure 2 : Evolution of inclinations for the theory of the first order (up) and the second order (down) on time interval 10^6 yr.

Results of integration. Ascending nodes



pl.	period	ampl.
J	49154	0.486
S	49154	1.017
U	433220	1.133
N	$1.89 \cdot 10^6$	0.803

pl.	period	ampl.
J	47822	0.483
S	47822	1.014
U	405482	1.192
N	$1.66 \cdot 10^6$	0.829

Figure 3 : Evolution of ascending nodes for the theory of the first order (up) and the second order (down) on interval 10^6 yr.

Conclusion

- The expansion of the Hamiltonian of four-planetary system into the Poisson series is constructed up to 2nd degree of a small parameter.
- The estimation accuracy of the Hamiltonian expansion is about 10^{-12} for Solar system's giant planets.
- We have constructed the averaged Hamiltonian and the generating function of the transformation up to 2nd degree of a small parameter.
- The motion equations and the functions for the change of variables were constructed in 2nd approximation too.
- The motion equations are integrated by Everhart method for the initial conditions corresponding orbital elements of Solar system's giant planets.
- The motion of the planets has an almost periodic character. Eccentricities and inclinations of the planetary orbits save small values. The short-term perturbations remain small over the entire period of the integration.
- In the future our semi-analytical motion theory will be used for the investigation of dynamic and resonant properties of extrasolar planetary systems.

Thank you for your attention!