The construction of averaged planetary motion theory by means computer algebra system Piranha

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The investigation of planetary systems dynamical evolution is one of important problems of celestial mechanics. In this work we consider the construction of averaged semi-analytical motion theory for a planetary system with four planets. We need to obtain motion equations in time-averaged orbital elements. The use of these elements allows to eliminate short-periodic perturbations in the planetary motion and to construct the motion theory for a long-time period.

For our purposes the non-averaged Hamiltonian of the four-planetary problem is written in Jacobi coordinates.

$$h = -\sum_{i=1}^{4} \frac{M_i \kappa_i^2}{2a_i} + \mu \times G m_0 \Big\{ \sum_{i=2}^{4} \frac{m_i (2\mathbf{r}_i \mathbf{R}_i + \mu R_i^2)}{r_i \tilde{R}_i (r_i + \tilde{R}_i)} - \sum_{i=1}^{4} \sum_{j=1}^{i-1} \frac{m_i m_j}{|\rho_i - \rho_j|} \Big\}. \quad (1)$$

Here

$$\mathbf{R}_i = \sum_{k=1}^{i} \frac{m_k}{\bar{m}_k} \mathbf{r}_k, \quad \tilde{R}_i = \sqrt{r_i^2 + 2\mu \mathbf{r}_i \mathbf{R}_i + \mu^2 R_i^2}, \quad (2)$$

and

$$|\rho_i - \rho_j| = \mathbf{r}_i - \mathbf{r}_j + \mu \sum_{k=j}^{i-1} \frac{m_k}{\bar{m}_k} \mathbf{r}_k, \quad (3)$$

where numbers i and j satisfy a condition $1 \leq j < i \leq 4$; ρ_k is the barycentric radius vector of k-th planet, $\mathbf{r}_k$ is Jacobi radius vector of the same planet; μm_k is the mass of the planet in items of the Sun mass m_0, $\bar{m}_k = 1 + \mu m_1 + \ldots + \mu m_k$, $M_i = m_i \bar{m}_{i-1}/\bar{m}_i$, $\kappa_i^2 = G m_0 \bar{m}_i / \bar{m}_{i-1}$ is the gravitational parameter and μ is the small parameter of the problem, which is equal to the ratio of the sum of planetary masses and the mass of the star. For instance, if we take into account the Solar system then the value of μ can take equal to 0.001.

The first sum in (1) is the undisturbed part of the Hamiltonian, which describes the Keplerian motion of planets around the Sun. The expression in figure brackets is the disturbing function. Double sum in (1) is the main part of the disturbing function, which describes the interaction between planets.

Further it is expanded into the Poisson series in orbital elements of the Poincare second system. This system has only one angular element – mean longitude. It allows to simplify an angular part of the series expansion. The elements of the second Poincare system are defined through classical Keplerian elements by the following way

$$L_i = M_i\sqrt{\kappa_i^2 a_i}, \quad \lambda_i = \Omega_i + \omega_i + l_i,$$

$$\xi_{1i} = \sqrt{2L_i(1-\sqrt{1-e_i^2})}\cos(\Omega_i+\omega_i),\ \xi_{2i} = \sqrt{2L_i\sqrt{1-e_i^2}(1-\cos I_i)}\cos\Omega_i,$$

$$\eta_{1i} = -\sqrt{2L_i(1-\sqrt{1-e_i^2})}\sin(\Omega_i+\omega_i),\ \eta_{2i} = -\sqrt{2L_i\sqrt{1-e_i^2}(1-\cos I_i)}\sin\Omega_i,$$

where M_i is normalized mass, κ_i^2 is normalized gravitational parameter, a_i – semi-major axis of the orbital ellipse, e_i – eccentricity of this ellipse, I_i – inclination of the orbital plane relative to the reference plane, quantities Ω_i, ω_i, l_i are longitude of the ascending node, argument of the pericenter and mean anomaly of the planet respectively.

The elements of second Poincare system are canonical and three pairs of these are canonical conjugated as the momentum and its the corresponding coordinate, namely L and λ, ξ_1 and η_1, ξ_2 and η_2.

The Hamiltonian of the planetary problem can be expanded into the Poisson series in the following form

$$h = h_0 + \mu h_1 = h_0 + \sum_{k,n} A_{kn} x^k \cos(n\lambda), \tag{4}$$

where h_0 is the undisturbed Hamiltonian, μh_1 is the disturbing function, A_{kn} is numerical coefficients, x^k is the product of Poincare elements with corresponding degrees, cosine is represent the angular part of the series, $n\lambda$ is the linear combination of mean longitudes of planets.

In our work the expansion of the Hamiltonian is constructed up to the second degree of the small parameter. The algorithm of the Hamiltonian expansion is described more detail in [1].

The averaged Hamiltonian of the four-planetary problem is constructed by the Hori-Deprit method. This averaging method based on using of Poisson brackets formalism and theory of Lie transformation. It is characterized by efficiency and very ease for the computer implementation. More detail see in [2].

Let us divide the variables of the problem into two parts – slow variables $x = (L, \xi_1, \eta_1, \xi_2, \eta_2)$ and fast λ. The rates of change for slow variables are proportionally the small parameter while the rates of change for fast variables are proportion to the mean motions. After averaging transformation with respect to the mean longitudes λ, the Hamiltonian is written in averaged slow variables X as the series of the small parameter

$$H(X) = H_0 + \sum_{m=1}^{\infty} \mu^m H_m(X), \tag{5}$$

where quantities H_m are obtained from the main equation of the Hori–Deprit method

$$H_m(X) = h_m + \sum \frac{1}{r!} \{T_{j_r}, \{\cdots, \{T_{j_1}, h_{j_0}\}\}\}. \tag{6}$$

The summation is over the domain $0 \leq j_0 \leq m-1$; $j_1, j_2, \cdots, j_r \geq 1$; $\sum_{s=0}^{k} j_s = m$; $1 \leq r \leq m$. The figure brackets is Poisson brackets with respect to the Poincare elements. h_m are items of not averaged Hamiltonian h, and the generating function of the transformation between osculating and averaging elements is defined as

$$T(X, \Lambda) = \sum_{m=1}^{\infty} \mu^m T_m(X, \Lambda). \tag{7}$$

Averaged motion equations can be obtained using Poisson brackets

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \{H, X\}, \quad \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \{H, \Lambda\}. \tag{8}$$

The transformation from osculating to averaged elements gives by functions for the change of variables u_m, v_m

$$X = x + \sum_{m=1}^{\infty} (-1)^m \mu^m u_m(x, \lambda), \quad u_m = \sum \frac{1}{r!} \{T_{j_r}, \{\cdots, \{T_{j_1}, X\}\}\} \tag{9}$$

$$\Lambda = \lambda + \sum_{m=1}^{\infty} (-1)^m \mu^m v_m(x, \lambda), \quad v_m = \sum \frac{1}{r!} \{T_{j_r}, \{\cdots, \{T_{j_1}, \Lambda\}\}\} \tag{10}$$

where the summation over the domain $j_1, j_2, \cdots, j_r \geq 1$; $\sum_{s=0}^{k} j_s = m$; $1 \leq r \leq m$.

All analytical transformations in our work are implemented by means of computer algebra system Piranha [3]. Piranha is an echeloned Poisson series processor. It is new, specified, high-efficient C++ code for analytical manipulations with different series. Piranha is freeware, object-oriented and cross-platform software. For the convenience Piranha has Python user-interface which is the set of some Python libraries. This program was written by Francesco Biscani from Heidelberg University, Germany.

Piranha can works with multivariable polynomials, Poisson series and echeloned Poisson series (Poisson series with denominators). It is possible to use real or rational types of series coefficients and powers of variables. In this work we used echeloned Poisson series with rational coefficients and powers that allows to eliminate rounding errors and provides arbitrary precision of resulting series.

In the process Piranha showed a high speed of analytical transformations and ability to work with the series of a very large number of terms (up to $10^8 - 10^9$ terms).

Finally we have applied our averaged motion theory to the investigation of orbital evolution of Solar system's giant planets. The results of numerical integration of the averaged motion equations for Sun - Jupiter - Saturn - Uranus - Neptune's system on a time interval of 10 billion years is considered. The obtained results show qualitative agreement with other motion theories.

References

[1] A.S. Perminov and E.D. Kuznetsov, *Expansion of the Hamiltonian of the planetary problem into the Poisson series in elements of the second Poincare system*, Solar System Research. **49**, 6, pp. 430-441 (2015).

[2] A.S. Perminov and E.D. Kuznetsov, *The Hori-Deprit method for averaged motion equations of the planetary problem in elements of the second Poincare system*, Solar System Research. **50**, 6, pp. 426-436 (2016).

[3] F. Biscani, *The Piranha computer algebra system*. https://github.com/bluescarni/piranha (2017).