Wave fronts and caustics of vector fields

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Consider a smooth parametrized plane curve $\mathcal{L}$. The wave front at the distance $|h|$ from the curve $\mathcal{L}$ is the set of points $\mathcal{L}_h$, each of which is at the distance $|h|$ from some point of the curve $\mathcal{L}$ in the direction of the normal to the curve $\mathcal{L}$ to the left of the direction of the path-tracing for $h > 0$, or to the right of the direction of the path-tracing for $h < 0$.

Singularities of the wave front of a smooth plane curve are well understood (see [1, 2]). However the proofs of the main results are distributed in various papers and often fairly complicated. For concrete curves (such as ellipse, for example [3]) explicit formulas are either absent, or derived with rather complicated proofs. However, if smooth plane curves are considered as integral curves of a smooth vector field, then simple explicit formulas for singularities of their wave fronts, asymptotics at these singular points, caustics and their singularities, etc., can be derived with little effort and simple means. All that is needed is to integrate the ODE system describing the vector field.

Consider a smooth planar vector field $\mathbf{V}$ without singularities in some domain $\mathcal{D} \subset \mathbb{R}^2$, i.e. let two smooth functions $p(x, y)$ and $q(x, y)$ be given such that $p^2 + q^2 \neq 0$ in the domain $\mathcal{D}$.

Let $\mathcal{L}$ be an integral curve of the vector field $\mathbf{V}$ passing through the point (x, y) and parametrized by a parameter t, i.e. the functions $x(t)$ and $y(t)$ satisfy the ODE system

$$dx/dt = \dot{x} = p(x, y), \quad dy/dt = \dot{y} = q(x, y). \tag{1}$$

At every point of the domain $\mathcal{D}$, the vectors $\mathbf{v} = (p, q)'$ and $\mathbf{w} = (-q, p)'$ are orthogonal, hence the wave front $\mathcal{L}_h$ (which may not belong to the domain $\mathcal{D}$) is also a parametrized curve with the same parameter t, i.e. $\mathcal{L}_h = (x_h(t), y_h(t))$, where

$$x_h = x - h\frac{q}{\sqrt{p^2 + q^2}}, \quad y_h = y + h\frac{p}{\sqrt{p^2 + q^2}}. \tag{2}$$

If we differentiate the equations (2) with respect to the parameter t, we obtain the ODE system for which the wave front $\mathcal{L}_h$ is an integral curve:

$$\dot{x}_h = p\,(1 - h\,k), \quad \dot{y}_h = q\,(1 - h\,k), \quad k = \frac{\dot{q}\,p - q\,\dot{p}}{(p^2 + q^2)^{3/2}}, \tag{3}$$

where k is the curvature (with a sign) of the curve $\mathcal{L}$ at the point t (i.e. at the point $(x, y) = (x(t), y(t))$). Here and further a *point* t corresponds to its own point on each curve, since they are all parametrized by the same parameter.

We note that all derivatives with respect to the parameter t are expressed through the derivatives with respect to the coordinates (x, y); for example, $\dot{p} = p\,\partial p/\partial x + q\,\partial p/\partial y$, etc.

We recall that absolute value of the curvature of the curve at a point is the reciprocal value or the radius of the closest fitting circle with the curve at the point. In this case the circle is given by the equation $(X - x_h)^2 + (Y - y_h)^2 = h^2$. We express Y through X (without loss of generality) and choose the semi-circle that has $dY/dX = q/p$ at the point $(X = x, Y = y)$. Equating the second derivatives $d^2Y/dX^2 = d^2y/dx^2$ at this point for the circle and the curve $\mathcal{L}$, we obtain the equation for the radius of the closest fitting circle.

Thus, the tangent line to the wave front (or its limit position at a singular point) is always parallel to the tangent line of the curve itself. Besides, the wave front $\mathcal{L}_h$ has singularities only at the points t where the value $1/h$ coincides with the curvature k of the curve $\mathcal{L}$. This is intuitively obvious, since the wave front $\mathcal{L}_h$ for the circle of radius h is the center of the circle.

Theorem 1 *Singularities of the wave front $\mathcal{L}_h$ coincide with singularities of the mapping $H\colon \mathcal{D} \to \mathbb{R}^2$ given by the formulas* (2).

Proof. $|DH| = 1 - h\dfrac{p^2\, q_x - q^2\, p_y + pq\,(q_y - p_x)}{(p^2 + q^2)^{3/2}} = 1 - h\,k.$

The systems of ODEs (1) and (3) should be solve concurrently. If the system (1) is integrated, then we have the wave front as a parametrized curve (2).

Theorem 2 *In generic case, a singular point of the wave front $\mathcal{L}_h$ is the vertex of a cusp. The two branches of the cusp have asymptotics of two semi-cubic parabolas.*

Proof. Parametrization of the values x_h, y_h in the neighborhood of the singular point of the wave front $\mathcal{L}_h$ does not have linear terms, hence the statement of the theorem is a corollary of the following (obvious) lemma.

Lemma 1 *Let the curve $\mathcal{A}$ be given parametrically:*

$$x = t^2 + a_3\, t^3 + a_4\, t^4 + \ldots, \qquad y = b_2\, t^2 + b_3\, t^3 + b_4\, t^4 + \ldots \tag{4}$$

then the curve $\mathcal{A}$ has two branches at the origin:

$$y_{1,2} = b_2\, x \pm (b_2\, a_3 - b_3)\, x^{3/2} + \frac{1}{2}\,(3\, b_2\, a_3^2 - 2\, b_2\, a_4 - 3\, b_3\, a_3 + 2\, b_4)\, x^2 \pm \ldots$$

Since $h = 1/k$ at singular points of the wave front $\mathcal{L}_h$, then the envelope of these singularities, i.e. the *caustic* $\mathcal{C}$ of all wave fronts $\mathcal{L}_h$, has the parametric representation

$$x_c = x - \frac{q}{k\sqrt{p^2 + q^2}}, \quad y_c = y + \frac{p}{k\sqrt{p^2 + q^2}}. \tag{5}$$

In particular, the caustic has asymptotes at the points t where the curvature of the curve $\mathcal{L}$ vanishes, for example, at the inflection points of the curve $\mathcal{L}$.

Theorem 3 *In generic case, the singular points of the caustic $\mathcal{C}$ have the same type as the singular points of the wave front, i.e. they are described by Theorem 2. In addition, the singular points of the caustic $\mathcal{C}$ are those and only those values of the parameter t where the curvature of the curve $\mathcal{L}$ has conditional extremum.*

Proof. Differentiating equalities (5) with respect to t, we obtain

$$\dot{x}_c = \frac{\dot{k}q}{k^2\sqrt{p^2+q^2}}, \quad \dot{y}_c = -\frac{\dot{k}p}{k^2\sqrt{p^2+q^2}}, \tag{6}$$

hence Lemma 1 is applicable.

Formulas (6) imply that the tangent to the caustic $\mathcal{C}$ at non-singular points is orthogonal to the tangent to the curve $\mathcal{L}$.

Remark 1 If an extremum of the curvature of the curve $\mathcal{L}$ at a singular point of the caustic $\mathcal{C}$ is not strict, then this is not generic case. If, for example, $\ddot{k} = 0$ at the singular point, while $d^3k/dt^3 \neq 0$, then under a small deformation of the curve $\mathcal{L}$, this singular point of the caustic can split into two ordinary points or vanish.

A singular point of the wave front $\mathcal{L}_h$ when $1/h$ is equal to the curvature of the curve $\mathcal{L}$ at the point of strict local extrema of the curvature is called the *extreme* point. Such points correspond to the singularities of the caustic, according to Theorem 3, and coincide with them in the plane, according to the formulas (2) and (5).

We will need the following obvious

Lemma 2 *Let the curve $\mathcal{B}$ be given parametrically:*

$$x = t^3 + a_4\,t^4 + a_5\,t^5 + \ldots, \quad y = b_3\,t^3 + b_4\,t^4 + b_5\,t^5 + \ldots$$

then the curve $\mathcal{B}$ has the following form at the origin:

$$y = b_3\,x - (b_3\,a_4 - b_4)\,x^{4/3} + \frac{1}{3}\,(4\,b_3\,a_4^2 - 3\,b_3\,a_5 - 4\,b_4\,a_4 + 3\,b_5)\,x^{5/3} + \ldots$$

Theorem 4 *In generic case, the asymptotics of the wave front $\mathcal{L}_h$ at extreme points is described by Lemma 2.*

Proof. Differentiating equalities (5) with respect to t, we obtain

$$\ddot{x}_h = \dot{p}\,(1 - h\,k) - hp\dot{k}, \quad \ddot{y}_h = \dot{q}\,(1 - h\,k) - hq\dot{k}.$$

We substitute $h = 1/k$ and obtain Lemma 2.

Theorem 5 *In generic case, the wave front $\mathcal{L}_h$ in the neighborhood of the extreme point is either a smooth curve* (*of the same smoothness as the curve* $\mathcal{L}$), *or it has two singularities, which collide with each other and vanish when h changes towards the value at the extreme point.*

Proof. The wave front passing through the extreme point has a tangent, according to Lemma 2, and this tangent is parallel to the tangent to the curve $\mathcal{L}$. The extreme point is the vertex of the cusp of the caustic (Theorem 3). The limit direction of tangents to the two branches of the cusp at the extreme point is orthogonal to the tangent to the wave front (see formula (6)). Hence the close wave fronts either intersect the caustic at two points or do not intersect it.

Finally, consider the case when the vector field depends on a real parameter λ. Then integral curves may deform under the change of the parameter λ, and bifurcations of the wave fronts and caustics are possible. Let, for example, the situation of Remark 1 take place. Then if we differentiate equalities (6) with respect to t and apply Lemma 2, we obtain that the bifurcation of the caustic under the change of λ is described by the behavior of the wave front in the neighborhood of the extreme point (with the fixed λ) when h changes, i.e. we have an analog of Theorem 5.

Theorem 6 *In the situation of Remark 1, the caustic $\mathcal{C}_\lambda$ in the neighborhood of the singular point is either a smooth curve* (*of the same smoothness as the curve* $\mathcal{L}_\lambda$), *or it has two singularities, which collide with each other and vanish when λ changes. In addition, the asymptotic of the caustic at the point where the two singularities vanish is described by Lemma 2.*

References

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