Non integrability of the colinear 3 and 4 body problem

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We prove the meromorphic non integrability of the colinear 3 and 4 body problem with positive masses. We prove moreover this this implies the non integrability of the 3 and 4 body problem in any dimension. This non integrabililty proof is based on Morales Ramis Theorem and in particular an algorithm on higher variational equations. The Morales Ramis Theorem put some conditions on the Galois group of variational equations near straight line orbit (the Galois group should be virtually abelian), corresponding in celestial mechanics to central configurations. In the colinear case, there are only few central configurations and a single real one. A quick analysis of the equation of central configuration shows that the quantities required to prove non integrability (eigenvalues of Hessian matrices of the potential at central configuration) are inaccessible using Groebner basis, at least in ≥ 4 bodies due to computer performance. We then present a new approach combining real algebraic geometry and higher variational analysis.

- Using the fact that the masses are real and the existence of a real central configuration thanks to Moulton Theorem, we bound the eigenvalues of the Hessian matrix at this central configuration. Such a bound seems to exist for any number of bodies.
- With this majoration and Morales Ramis integrabililty criterion, we come down to study 26 cases. Each case is analyzed through higher variational equations.

Higher variational equations are put on the form of a triangular linear differential equation

$$\dot{X} = A(t)X \qquad A \in M_n(\mathbb{C}(t))$$

with A upper triangular and A could depend polynomially on some parameters a outside the diagonal (and so avoiding indicial polymials with parameters). These parameters correspond to higher order derivatives of the potential at the central configuration. The Galois group of such equations can be studied by searching rational solutions of this system. Several improvements in performance using the structure of A are necessary due to the size of $A \geq 100$ in worst cases. The output of the algorithm is an ideal on the parameters for which the Galois group is abelian. Again computer performance is not enough to test these conditions, but we discover that pushing variational equations up to order 5 produces algebraic sets without any real points. Thus variational equations of order 5 cannot have a virtually abelian Galois group if the derivatives of the potential are real, which is the case because masses and the central configuration are real.