On Dependence of Nonperturbative Contributions in $\bar{\alpha}_s(q^2)$ on an Initial Approximation of erturbation Theory in an Analytic Approach to QCD

Aleksey I. Alekseev

Institute for High Energy Physics, 142280 Protvino, Moscow Region
Russia

Abstract

In the framework of analytic approach to QCD, which has been recently intensively developed, the dependence of nonperturbative contributions in a running coupling of strong interaction on initial perturbative approximation to the 3-loop order is studied. The nonperturbative contributions are obtained in an explicit form. In the ultraviolet region they are shown to be represented in the form of the expansion in the inverse powers of Euclidean momentum squared. The expansion coefficients are calculated for different numbers of active quark flavors n_f and for different numbers of loops taken into account. For all n_f of interest it is shown that the 2-loop order and the 3-loop order corrections result in partial compensation of the 1-loop order leading in the ultraviolet region nonperturbative contribution.

It is widely believed that unphysical singularities of the perturbation theory in the infrared region of QCD should be canceled by the nonperturbative contributions. The nonperturbative contributions arise quite naturally in an analytic approach [1] to QCD. The so-called "analyticization procedure" is used in this approach. The main purpose of this procedure is to remove nonphysical singularities from approximate (perturbative) expressions for the Green functions of QFT. The idea of the procedure goes back to Refs. [2, 3] devoted to the ghost pole problem in QED. The foundation of the procedure is the principle of summation of imaginary parts of the perturbation theory terms. Then, the Källen – Lehmann spectral representation results in the expressions without nonphysical singularities. In recent papers [4, 5] it is suggested to solve the ghost pole problem in QCD demanding the $\bar{\alpha}_s(q^2)$ be analytical in q^2 (to compare with dispersive approach [6]). As a result, instead of the one-loop expression $\bar{\alpha}_s^{(1)}(q^2) = (4\pi/b_0)/\ln(q^2/\Lambda^2)$ taking into account the leading logarithms and having the ghost pole at $q^2 = \Lambda^2$ (q^2 is the Euclidean momentum squared), one obtains the expression

$$\bar{\alpha}_{an}^{(1)}(q^2) = \frac{4\pi}{b_0}\left[\frac{1}{\ln(q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - q^2}\right]. \tag{1}$$

Eq. (1) is an analytic function in the complex q^2-plane with a cut along the negative real semiaxis. The pole of the perturbative running coupling at $q^2 = \Lambda^2$ is canceled by the nonperturbative contribution ($\Lambda^2 \mid_{g^2\to 0} \simeq \mu^2 \exp\{-(4\pi)^2/(b_0 g^2)\}$) and the value $\bar{\alpha}_{an}^{(1)}(0) = 4\pi/b_0$ appeared finite and independent of Λ. The most important feature of the "analyticization procedure" discovered [4, 5] is the stability property of the value of the "analytically improved" running coupling constant at zero with respect to high corrections, $\bar{\alpha}_{an}^{(1)}(0) = \bar{\alpha}_{an}^{(2)}(0) = \bar{\alpha}_{an}^{(3)}(0)$. This property provides the high corrections stability of $\bar{\alpha}_{an}(q^2)$ in the whole infrared region.

The 1-loop order nonperturbative contribution in Eq. (1) can be presented as convergent at $q^2 > \Lambda^2$ constant signs series in the inverse powers of the momentum squared. For "standard" as well as for the iterative 2-loop perturbative input the nonperturbative contributions in the analytic running coupling are calculated explicitly in Ref. [7]. In the ultraviolet region the nonperturbative contributions can be also represented as a series in inverse powers of momentum squared. In this paper we extract in an explicit form the nonperturbative contributions to $\bar{\alpha}_{an}(q^2)$ up to the 3-loop order in the analytic approach to QCD, and investigate their ultraviolet behavior. To handle the singularities originating from the perturbative input, we develop here the method which is more general than that of Ref. [7].

According to the definition the analytic running coupling is obtained by the integral representation

$$a_{an}(x) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{x+\sigma}\rho(\sigma), \tag{2}$$

where the spectral density $\rho(\sigma) = \mathrm{Im}\, a_{an}(-\sigma - i0)$. It is seen that dispersively-modified coupling of form (2) has an analytical structure which is consistent with causality. According to the analytic approach to QCD we adopt that $\mathrm{Im}\, a_{an}(-\sigma - i0) = \mathrm{Im}\, a(-\sigma - i0)$ where $a(x)$ is an appropriately normalized perturbative running coupling.

The behavior of the QCD running coupling $\alpha_s(\mu^2)$ is defined by the renormalization group equation

$$\mu \frac{\partial \alpha_s(\mu^2)}{\partial \mu} = \beta(\alpha_s) = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \beta_2 \alpha_s^4 + ..., \tag{3}$$

where the coefficients

$$\begin{gathered} \beta_0 = -\frac{1}{2\pi} b_0, \;\; b_0 = 11 - \frac{2}{3} n_f, \\ \beta_1 = -\frac{1}{4\pi^2} b_1, \;\; b_1 = 51 - \frac{19}{3} n_f, \\ \beta_2 = -\frac{1}{64\pi^3} b_2, \;\; b_2 = 2857 - \frac{5033}{9} n_f + \frac{325}{27} n_f^2. \end{gathered} \tag{4}$$

The first two coefficients β_0, β_1 do not depend on the renormalization scheme choice. Here n_f is the number of active quark flavors. The standard three-loop solution of Eq. (3) is written in the form of expansion in inverse powers of logarithms [8]

$$\begin{aligned} \alpha_s(\mu^2) = \frac{4\pi}{b_0 \ln(\mu^2/\Lambda^2)} \Bigg[1 - \frac{2b_1}{b_0^2} \frac{\ln[\ln(\mu^2/\Lambda^2)]}{\ln(\mu^2/\Lambda^2)} + \\ + \frac{4b_1^2}{b_0^4 \ln^2(\mu^2/\Lambda^2)} \times \left(\left(\ln[\ln(\mu^2/\Lambda^2)] - \frac{1}{2} \right)^2 + \frac{b_2 b_0}{8 b_1^2} - \frac{5}{4} \right) \Bigg]. \end{aligned} \tag{5}$$

Let us introduce the function $a(x) = (b_0/4\pi)\bar{\alpha}_s(q^2)$, where $x = q^2/\Lambda^2$. Then, instead of (5) one can write

$$a(x) = \frac{1}{\ln x} - b \frac{\ln(\ln x)}{\ln^2 x} + b^2 \left(\frac{\ln^2(\ln x)}{\ln^3 x} - \frac{\ln(\ln x)}{\ln^3 x} + \frac{\kappa}{\ln^3 x} \right), \tag{6}$$

where the coefficients b and κ are equal to

$$b = \frac{2b_1}{b_0^2} = \frac{102 - \frac{38}{3} n_f}{(11 - \frac{2}{3} n_f)^2}, \quad \kappa = \frac{b_0 b_2}{8 b_1^2} - 1. \tag{7}$$

At $n_f = 3$ $b_0 = 9$, and $b = 64/81 \simeq 0.7901$, $\kappa \simeq 0.4147$. At $x \simeq 1$ the perturbative running coupling is singular. At large x the 1-loop term of Eq. (6) defines the ultraviolet behavior of $a(x)$ but for small x the behavior of the running coupling depends on the approximation we adopt, and at $x = 1$ there are singularities of a different analytical structure. Namely, at $x \simeq 1$

$$a^{(1)}(x) \simeq \frac{1}{x-1}, \;\; a^{(2)}(x) \simeq -\frac{b}{(x-1)^2} \ln(x-1), \;\; a^{(3)}(x) \simeq \frac{b^2}{(x-1)^3} \ln^2(x-1). \tag{8}$$

This is not an obstacle for the analytic approach which removes this nonphysical singularities. By making the analytic continuation of Eq. (6) into the Minkowski space $x = -\sigma - i0$, one obtains

$$\begin{aligned} a(-\sigma - i0) = \frac{1}{\ln \sigma - i\pi} - \frac{b}{(\ln \sigma - i\pi)^2} \ln(\ln \sigma - i\pi) + b^2 \Bigg\{ \frac{\ln^2(\ln \sigma - i\pi)}{(\ln \sigma - i\pi)^3} - \\ - \frac{\ln(\ln \sigma - i\pi)}{(\ln \sigma - i\pi)^3} + \frac{\kappa}{(\ln \sigma - i\pi)^3} \Bigg\}. \end{aligned} \tag{9}$$

Function $a(x)$ in Eq. (6) is regular and real for real $x > 1$. So, to find the spectral density $\rho(\sigma)$ we can use the reflection principle $(a(x))^* = a(x^*)$ where x is considered as a complex variable. Then

$$\rho(\sigma) = \frac{1}{2i} \left(a(-\sigma - i0) - a(-\sigma + i0) \right). \tag{10}$$

By the change of variable of the form $\sigma = \exp(t)$, the analytical expression is derived from (2), (9), (10) as follows:

$$a_{an}(x) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} dt\, \frac{e^t}{x+e^t} \times \left\{ \frac{1}{t-i\pi} - \frac{1}{t+i\pi} - \right.$$

$$-b\left[\frac{\ln(t-i\pi)}{(t-i\pi)^2} - \frac{\ln(t+i\pi)}{(t+i\pi)^2}\right] + b^2\left[\frac{\ln^2(t-i\pi)}{(t-i\pi)^3} - \frac{\ln^2(t+i\pi)}{(t+i\pi)^3} - \right.$$

$$\left.\left. -\frac{\ln(t-i\pi)}{(t-i\pi)^3} + \frac{\ln(t+i\pi)}{(t+i\pi)^3} + \frac{\kappa}{(t-i\pi)^3} - \frac{\kappa}{(t+i\pi)^3}\right]\right\}. \tag{11}$$

Let us see what the singularities of the integrand of (11) in the complex t-plane are. First of all the integrand has simple poles at $t = \ln x \pm i\pi(1+2n)$, $n = 0, 1, 2, \ldots$. All the residues of the function $\exp(t)/(x+\exp(t))$ at these points are equal to unity. Apart from these poles the integrand of (11) has simple poles at $t = \pm i\pi$, the third order poles and logarithmic type branch points which coincide with the second order and third order poles. Let us cut the complex t-plane in a standard way, $t = \pm i\pi - \lambda$, with λ being the real parameter varying from 0 to ∞. The integrand in (11) multiplied by t goes to zero at $\mid t \mid \to \infty$. That allows one to append the integration by the arch of the "infinite" radius without affecting the value of the integral. Close the integration contour C_1 in the upper half-plane of the complex variable t excluding the singularity at $t = i\pi$. In this case an additional contribution emerges due to the integration along the sides of the cut and around the singularities at $t = i\pi$. The corresponding contour we denote as C_2.

Let us turn to the integration along contour C_1. For the integrand of Eq. (11) which we denote as $F(t)$ the residues at $t = \ln(x) + i\pi(1+2n)$, $n = 0, 1, 2, \ldots$ are as follows

$$\mathrm{Res}\, F(t)\mid_{t=\ln(x)+i\pi(1+2n)} = \frac{1}{\ln(x)+2\pi i n} - \frac{1}{\ln(x)+2\pi i(n+1)} -$$

$$-b\left[\frac{\ln(\ln(x)+2\pi i n)}{(\ln(x)+2\pi i n)^2} - \frac{\ln(\ln(x)+2\pi i(n+1))}{(\ln(x)+2\pi i(n+1))^2}\right] + b^2\left[\frac{\ln^2(\ln(x)+2\pi i n)}{(\ln(x)+2\pi i n)^3} - \frac{\ln^2(\ln(x)+2\pi i(n+1))}{(\ln(x)+2\pi i(n+1))^3} - \right.$$

$$\left. - \frac{\ln(\ln(x)+2\pi i n)}{(\ln(x)+2\pi i n)^3} + \frac{\ln(\ln(x)+2\pi i(n+1))}{(\ln(x)+2\pi i(n+1))^3} + \frac{\kappa}{(\ln(x)+2\pi i n)^3} - \frac{\kappa}{(\ln(x)+2\pi i(n+1))^3}\right]. \tag{12}$$

By using the residue theorem one readily obtains the contribution $\Delta(x)$ to integral (11) from the integration along contour C_1. It reads

$$\Delta(x) = \frac{1}{2\pi i}\int\limits_{C_1} F(t)\, dt = \sum_{n=0}^{\infty} \mathrm{Res}\, F\left(t = \ln(x) + i\pi(1+2n)\right) =$$

$$= \frac{1}{\ln x} - b\frac{\ln(\ln x)}{\ln^2 x} + b^2\left(\frac{\ln^2(\ln x)}{\ln^3 x} - \frac{\ln(\ln x)}{\ln^3 x} + \frac{\kappa}{\ln^3 x}\right). \tag{13}$$

We can see that this contribution is exactly equal to initial Eq. (6). Therefore, we call it a perturbative part of $a_{an}(x)$, $a^{pt}(x) = \Delta(x)$. The remaining contribution of the integral along contour C_2 can be naturally called a nonperturbative part of $a_{an}(x)$,

$$a_{an}(x) = a^{pt}(x) + a_{an}^{npt}(x). \tag{14}$$

Let us turn to the calculation of $a_{an}^{npt}(x)$. We can omit the terms of the integrand in Eq. (11) which have no singularities at $t = i\pi$. Then, we have

$$a_{an}^{npt}(x) = \frac{1}{2\pi i}\int\limits_{C_2} dt\, \frac{e^t}{x+e^t} \times \left\{\frac{1}{t-i\pi} - b\frac{\ln(t-i\pi)}{(t-i\pi)^2} + \right.$$

$$\left. + b^2\left[\frac{\ln^2(t-i\pi)}{(t-i\pi)^3} - \frac{\ln(t-i\pi)}{(t-i\pi)^3} + \frac{\kappa}{(t-i\pi)^3}\right]\right\}. \tag{15}$$

Let us change the variable $t = z + i\pi$ and introduce the function

$$f(z) = \frac{1}{1 - x\exp(-z)}. \tag{16}$$

Then, we can rewrite Eq. (15) in the form

$$a_{an}^{npt}(x) = \frac{1}{2\pi i}\int\limits_C dz\, f(z)\left\{\frac{1}{z} - b\frac{\ln(z)}{z^2} + b^2\frac{\ln^2(z)}{z^3} - b^2\frac{\ln(z)}{z^3} + \frac{\kappa b^2}{z^3}\right\}. \tag{17}$$

The cut in the complex z-plane goes now from zero to $-\infty$. Starting from $z = -\infty - i0$ contour C goes along the lower side of the cut, then goes around the origin, and next it goes further along the upper side of the cut to $z = -\infty + i0$. Here we consider x as a real variable, $x > 1$. Then, contour C can be chosen in such a way that it does not envelop "superfluous" singularities and the conditions used in Appendix to find the corresponding integrals be satisfied. Function (16) with its derivatives decrease exponentially at $z \to -\infty$, therefore, we shall omit the boundary terms in formulas given in Appendix. We shall need further the explicit expressions for the derivatives of $f(z)$, which read

$$f'(z) = -\frac{x\exp(-z)}{(1 - x\exp(-z))^2},\quad f''(z) = \frac{x\exp(-z)(1 + x\exp(-z))}{(1 - x\exp(-z))^3},$$

$$f'''(z) = -\frac{x\exp(-z)}{(1 - x\exp(-z))^4}\left(1 + 4x\exp(-z) + x^2\exp(-2z)\right). \tag{18}$$

¿From Eq. (17) one can obtain

$$a_{an}^{npt}(x) = -\frac{1}{2\pi i}\int\limits_C dz\,\left\{f'(z)\ln(z) - bf''(z)\left(\ln(z) + \frac{1}{2}\ln^2(z)\right) + \right.$$

$$\left. + b^2 f'''(z)\left[\left(1 + \frac{1}{2}\kappa\right)\ln(z) + \frac{1}{2}\ln^2(z) + \frac{1}{6}\ln^3(z)\right]\right\}. \tag{19}$$

Taking into account that function $f(z)$ with its derivatives is regular at real negative semiaxis of z, we can rewrite equation (19) in the form

$$a_{an}^{npt}(x) = -\int\limits_0^{-\infty} du\,\left[f'(u)\Delta_1(u) - bf''(u)\left(\Delta_1(u) + \frac{1}{2}\Delta_2(u)\right) + \right.$$

$$\left. + b^2 f'''(u)\left(\left(1 + \frac{1}{2}\kappa\right)\Delta_1(u) + \frac{1}{2}\Delta_2(u) + \frac{1}{6}\Delta_3(u)\right)\right], \tag{20}$$

where u is real, $u < 0$ and $\Delta_i(u)$ are discontinuities of the logarithms

$$\begin{aligned}
\Delta_1(u) &= \frac{1}{2\pi i}\left(\ln(u + i0) - \ln(u - i0)\right) = 1,\\
\Delta_2(u) &= \frac{1}{2\pi i}\left(\ln^2(u + i0) - \ln^2(u - i0)\right) = 2\ln(-u),\\
\Delta_3(u) &= \frac{1}{2\pi i}\left(\ln^3(u + i0) - \ln^3(u - i0)\right) = 3\ln^2(-u) - \pi^2.
\end{aligned} \tag{21}$$

Let us introduce the variable $\sigma = \exp(u)$. From Eqs. (18), (20), (21) we obtain

$$a_{an}^{npt}(x) = -x\int\limits_0^1 d\sigma\,\left\{\frac{1}{(x - \sigma)^2} - b\left[1 + \ln\left(-\ln(\sigma)\right)\right]\frac{x + \sigma}{(x - \sigma)^3} + \right.$$

$$\left. + b^2\left[1 - \frac{\pi^2}{6} + \frac{\kappa}{2} + \ln\left(-\ln(\sigma)\right) + \frac{1}{2}\ln^2\left(-\ln(\sigma)\right)\right]\frac{x^2 + 4x\sigma + \sigma^2}{(x - \sigma)^4}\right\}. \tag{22}$$

Integrating the terms of Eq. (22) independent of logarithms one can obtain

$$a_{an}^{npt}(x)=\frac{1}{1-x}+b\left\{\frac{x}{(1-x)^2}+x\int\limits_0^1 d\sigma\,\ln(-\ln(\sigma))\frac{x+\sigma}{(x-\sigma)^3}\right\}+$$

$$+b^2\left\{\left(1+\frac{\kappa}{2}-\frac{\pi^2}{6}\right)\frac{x(1+x)}{(1-x)^3}-x\int\limits_0^1 d\sigma\left[\ln(-\ln(\sigma))+\frac{1}{2}\ln^2(-\ln(\sigma))\right]\frac{x^2+4x\sigma+\sigma^2}{(x-\sigma)^4}\right\}. \qquad (23)$$

This formula gives the nonperturbative contributions in an explicit form and is convenient for numerical study.

Let us turn to the ultraviolet behavior of the nonperturbative contributions. Expanding Eq. (23) in inverse powers of x we have

$$a_{an}^{npt}(x)=\sum_{n=1}^{\infty}\frac{c_n}{x^n}, \qquad (24)$$

where

$$c_n=-1+bn\left[1+n\int\limits_0^1 d\sigma\,\sigma^{n-1}\ln\left(-\ln(\sigma)\right)\right]-$$

$$-b^2n^2\left\{1-\frac{\pi^2}{6}+\frac{\kappa}{2}+n\int\limits_0^1 d\sigma\,\sigma^{n-1}\left[\ln\left(-\ln(\sigma)\right)+\frac{1}{2}\ln^2\left(-\ln(\sigma)\right)\right]\right\}. \qquad (25)$$

Making the change of variable $\sigma=\exp(-t)$ and integrating [9], [10] over t, one can find

$$\int\limits_0^1 d\sigma\,\sigma^{n-1}\ln\left(-\ln(\sigma)\right)=\int\limits_0^\infty dt\,e^{-nt}\ln(t)=-\frac{1}{n}\left(\ln(n)+\gamma\right), \qquad (26)$$

$$\int\limits_0^1 d\sigma\,\sigma^{n-1}\ln^2\left(-\ln(\sigma)\right)=\int\limits_0^\infty dt\,e^{-nt}\ln^2(t)=\frac{1}{n}\left[\left(\ln(n)+\gamma\right)^2+\frac{\pi^2}{6}\right]. \qquad (27)$$

¿From Eqs. (25), (26), (27) we finally have

$$c_n=-1+bn\left(1-\gamma-\ln(n)\right)-\frac{1}{2}b^2n^2\left[1-\frac{\pi^2}{6}+\kappa+\left(1-\gamma-\ln(n)\right)^2\right]. \qquad (28)$$

Here γ is the Euler constant, $\gamma\simeq 0.5772$. We can see from Eq. (28) that power series (24) is uniformly convergent at $x>1$ and its convergence radius is equal to unity. To see the details of the coefficients c_n behavior we give the results of the numerical study in Table I. The coefficients c_n depend on n, n_f and the number of loops taken into account. The 1-loop order contribution to c_n equals -1 for all n and n_f. The 2-loop order corrections to c_n decreases monotonously with increasing n for $n_f=0,3,4,5,6$. In this case the correction to the leading term ($n=1$) is positive (it varies from 0.36 at $n_f=0$ to 0.22 at $n_f=6$), whereas for the next terms the corrections are negative. The 3-loop order corrections to c_n increases at first and then monotonously decreases with increasing n for all n_f we consider.

In the ultraviolet region ($x\gg 1$) the nonperturbative contributions are determined by the first term of the series (24). At $n_f\le 4$ the 3-loop order corrections change insignificantly the 2-loop order results for nonperturbative contributions. At $n_f=5$ the 3-loop order correction is 2.4 times less than the 2-loop order correction and at $n_f=6$ it exceeds the 2-loop order correction. We also note that with increasing n_f the decrease of the 2-loop corrections is equilibrated to some excess by the increase of the 3-loop corrections.

It seems to be most interesting that an account for the 2-loop order corrections results in some compensation of the 1-loop order leading at large x term of the form $1/x$ and at 3-loop order compensation remains valid. We can see from Table I that at $n_f\ge 3$ the 3-loop order $|c_1|$ is somewhat less than the

2-loop order $|c_1|$. However, it apparently does not mean that in an analytic approach the high-loop corrections lead to the total compensation of nonperturbative contributions in the ultraviolet region.

I am indebted to B.A. Arbuzov, V.A. Petrov, V.E. Rochev for useful discussions. This work has been supported in part by RFBR under Grants No. 99-01-00091, No. 98-02-16690.

Appendix

We give here the identities we need in our computations which can be obtained by means of an integration by parts. Let function $f(z)$ of complex variable z is regular in some domain D where $z = 0 \in D$. Dealing with the singularities of the integrands at the origin of the pole type coinciding with the logarithmic type branch points, we cut domain D along real negative semiaxis. Then, for any contour C in cut domain $\tilde{D}$, which goes from $z_1 \neq 0$ to $z_2 \neq 0$, one can find

$$\int\limits_C \frac{dz}{z} f(z) = -\int\limits_C dz\, \ln(z) f'(z) + \ln(z) f(z)\Big|_{z_1}^{z_2},$$

$$\int\limits_C \frac{dz}{z^3} f(z) = -\frac{1}{2}\int\limits_C dz\, \ln(z) f'''(z) + \left\{-\frac{1}{2z^2} f(z) - \frac{1}{2z} f'(z) + \frac{1}{2}\ln(z) f''(z)\right\}\Big|_{z_1}^{z_2},$$

$$\int\limits_C \frac{dz}{z^2} \ln(z) f(z) = -\int\limits_C dz\, \left(\ln(z) + \frac{1}{2}\ln^2(z)\right) f''(z) + \left\{-\frac{1}{z} f(z) - \frac{\ln(z)}{z} f(z) + \ln(z) f'(z) + \right.$$
$$\left. + \frac{1}{2}\ln^2(z) f'(z)\right\}\Big|_{z_1}^{z_2},$$

$$\int\limits_C \frac{dz}{z^3} \ln(z) f(z) = -\int\limits_C dz\, \left(\frac{3}{4}\ln(z) + \frac{1}{4}\ln^2(z)\right) f'''(z) + \left\{-\frac{1}{4z^2} f(z) - \frac{3}{4z} f'(z) - \frac{\ln(z)}{2z^2} f(z) - \right.$$
$$\left. - \frac{\ln(z)}{2z} f'(z) + \frac{3\ln(z)}{4} f''(z) + \frac{\ln^2(z)}{4} f''(z)\right\}\Big|_{z_1}^{z_2},$$

$$\int\limits_C \frac{dz}{z^3} \ln^2(z) f(z) = -\int\limits_C dz\, \left(\frac{7}{4}\ln(z) + \frac{3}{4}\ln^2(z) + \frac{1}{6}\ln^3(z)\right) f'''(z) + \left\{-\frac{1}{4z^2} f(z) - \frac{7}{4z} f'(z) - \frac{\ln(z)}{2z^2} f(z) - \right.$$
$$\left. - \frac{3\ln(z)}{2z} f'(z) + \frac{7\ln(z)}{4} f''(z) - \frac{\ln^2(z)}{2z^2} f(z) - \frac{\ln^2(z)}{2z} f'(z) + \frac{3\ln^2(z)}{4} f''(z) + \frac{\ln^3(z)}{6} f''(z)\right\}\Big|_{z_1}^{z_2}.$$

References

[1] I.L. Solovtsov and D.V. Shirkov, Theor. Math. Phys. 120 (1999) 1210

[2] P.J. Redmond, Phys. Rev. 112 (1958) 1404

[3] N.N. Bogolubov, A.A. Logunov, D.V. Shirkov, Sov. Phys. JETP 37 (1959) 805

[4] D.V. Shirkov, I.L. Solovtsov, JINR Rapid Comm. 2[76]-96 (1996) 5

[5] D.V. Shirkov, I.L. Solovtsov, Phys. Rev. Lett. 79 (1997) 1209

[6] Yu.L. Dokshitzer, G. Marchesini, B.R. Webber, Nucl. Phys. B469 (1996) 93

[7] A.I. Alekseev, Phys. Rev. D61 (2000) 114005

[8] Particle Data Group, Eur. Phys. J. C 15 (2000) 85

[9] H. Bateman and A. Erdelyi, Higher transcendental functions, Vol. 1, New York (McGraw-Hill 1953)

[10] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, Integrals and Series, Vol. 1, New York, NY (Gordon and Breach, 1986)

Table 1: The dependence of c_n on n and n_f for 1-loop, 2-loop and 3-loop cases.

	n	c_n^{1-loop}	Δ_{2-loop}	Δ_{3-loop}	c_n^{2-loop}	c_n^{3-loop}
$n_f = 0$	1	-1.0	0.35640	-0.01568	-0.64360	-0.65929
	2	-1.0	-0.45582	0.08741	-1.45582	-1.36841
	3	-1.0	-1.70912	-1.03012	-2.70912	-3.73924
	4	-1.0	-3.24886	-4.51236	-4.24886	-8.76122
	5	-1.0	-5.00160	-11.31238	-6.00160	-17.31398
	6	-1.0	-6.92407	-22.24971	-7.92407	-30.17378
	7	-1.0	-8.98770	-38.04599	-9.98770	-48.03369
	8	-1.0	-11.17217	-59.34790	-12.17217	-71.52007
	9	-1.0	-13.46228	-86.74274	-14.46228	-101.20502
	10	-1.0	-15.84626	-120.76945	-16.84626	-137.61571
$n_f = 3$	1	-1.0	0.33405	0.01608	-0.66595	-0.64987
	2	-1.0	-0.42724	0.19623	-1.42724	-1.23101
	3	-1.0	-1.60196	-0.63626	-2.60196	-3.23823
	4	-1.0	-3.04517	-3.48652	-4.04517	-7.53168
	5	-1.0	-4.68801	-9.19185	-5.68801	-14.87987
	6	-1.0	-6.48996	-18.47225	-7.48996	-25.96221
	7	-1.0	-8.42420	-31.96169	-9.42420	-41.38590
	8	-1.0	-10.47171	-50.22832	-11.47171	-61.70003
	9	-1.0	-12.61824	-73.78810	-13.61824	-87.40634
	10	-1.0	-14.85275	-103.11452	-15.85275	-118.96726
$n_f = 4$	1	-1.0	0.31252	0.04949	-0.68748	-0.63799
	2	-1.0	-0.39970	0.31340	-1.39970	-1.08630
	3	-1.0	-1.49872	-0.23818	-2.49872	-2.73690
	4	-1.0	-2.84891	-2.48499	-3.84891	-6.33389
	5	-1.0	-4.38587	-7.15989	-5.38587	-12.54576
	6	-1.0	-6.07168	-14.89306	-7.07168	-21.96474
	7	-1.0	-7.88126	-26.23939	-8.88126	-35.12065
	8	-1.0	-9.79681	-41.69613	-10.79681	-52.49294
	9	-1.0	-11.80500	-61.71490	-12.80500	-74.51990
	10	-1.0	-13.89549	-86.71011	-14.89549	-101.60561
$n_f = 5$	1	-1.0	0.27813	0.11653	-0.72187	-0.60535
	2	-1.0	-0.35571	0.55755	-1.35571	-0.79817
	3	-1.0	-1.33377	0.50736	-2.33377	-1.82641
	4	-1.0	-2.53536	-0.73077	-3.53536	-4.26613
	5	-1.0	-3.90317	-3.73728	-4.90317	-8.64045
	6	-1.0	-5.40344	-9.01126	-6.40344	-15.41470
	7	-1.0	-7.01386	-16.99218	-8.01386	-25.00604
	8	-1.0	-8.71859	-28.07387	-9.71859	-37.79246
	9	-1.0	-10.50576	-42.61401	-11.50576	-54.11977
	10	-1.0	-12.36618	-60.94080	-13.36618	-74.30698
$n_f = 6$	1	-1.0	0.22433	0.25378	-0.77567	-0.52189
	2	-1.0	-0.28692	1.07460	-1.28692	-0.21231
	3	-1.0	-1.07581	1.93179	-2.07581	-0.14402
	4	-1.0	-2.04500	2.37205	-3.04500	-0.67296
	5	-1.0	-3.14826	2.01775	-4.14826	-2.13052
	6	-1.0	-4.35837	0.54419	-5.35837	-4.81418
	7	-1.0	-5.65732	-2.33454	-6.65732	-8.99186
	8	-1.0	-7.03234	-6.87466	-8.03234	-14.90700
	9	-1.0	-8.47386	-13.30888	-9.47386	-22.78274
	10	-1.0	-9.97445	-21.85073	-10.97445	-32.82518