Gauge Invariant Regularisation in the ERG Approach

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Abstract

A gauge invariant regularisation which can be used for non-perturbative treatment of Yang-Mills theories within the exact renormalization group approach is constructed. It consists of a spontaneously broken $SU(N|N)$ super-gauge extension of the initial Yang-Mills action supplied with covariant higher derivatives. We demonstrate that the extended theory in four dimensions is perturbatively ultraviolet finite and argue that it has a sensible limit when the regularisation cutoff is removed.

1 Introduction

The exact renormalization group (ERG), which is the broadly accepted name for the continuous version of Wilson's renormalization group [1], is a quite powerful tool for studies in quantum field theory (see, for example, refs [2, 3, 4, 5]). It possesses some important features which motivate the work devoted to both further developments of the ERG formalism and to practical computations by this method. One of the advantageous features it offers is that it allows non-perturbative (though approximate in practice) studies and calculations. Another is that within the ERG approach almost all approximations preserve a crucial property of quantum field theory, namely the existence of the continuum limit. It also gives rise to the possibility of studying (at least within some approximation) the full parameter space of non-perturbative quantum field theories, its fixed points, continuum limits, etc. The challenge is to understand the low energy limit of QCD within the ERG approach (for some preliminary work on this see, for example, [6]).

The main ingredient of the ERG approach is an ERG equation. It determines the effective action (running action), S_Λ, as a function of the scale Λ which also plays the rôle of the effective momentum cutoff. Let us consider a scalar theory in D dimensions for simplicity. The effective action can be written as the sum of an (effective) action of interaction, $S_\Lambda^{int}[\phi]$, and the kinetic term, $\hat{S}_\Lambda[\phi]$:

$$S_\Lambda[\phi] = \hat{S}_\Lambda[\phi] + S_\Lambda^{int}[\phi].$$

To regularise the theory we modify the propagator in the momentum space, $1/p^2$, to $c(p^2/\Lambda^2)/p^2$, where $c(p^2/\Lambda^2)$ is a (smooth) ultraviolet cutoff profile satisfying $c(0) = 1$ so that low energies are unaltered, and $c(z) \to 0$ as $z \to \infty$ sufficiently fast so that all Feynman diagrams are ultraviolet regulated.
Introducing the shorthand

$$f \cdot W \cdot g := \int d^D x\, f(x) W\left(-\frac{\partial^2}{\Lambda^2}\right) g(x), \tag{1}$$

we write the regularised kinetic term as

$$\hat{S}_\Lambda[\phi] = \frac{1}{2}\partial_\mu\phi \cdot c^{-1} \cdot \partial^\mu\phi = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\, \phi(-p) p^2 c^{-1}\left(\frac{p^2}{\Lambda^2}\right)\phi(p). \tag{2}$$

Polchinski's version of the ERG equation [7] is

$$\Lambda\frac{\partial S_\Lambda^{int}}{\partial\Lambda} = -\frac{1}{\Lambda^2}\frac{\delta S_\Lambda^{int}}{\delta\phi} \cdot c' \cdot \frac{\delta S_\Lambda^{int}}{\delta\phi} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\phi} \cdot c' \cdot \frac{\delta S_\Lambda^{int}}{\delta\phi}. \tag{3}$$

Being supplied with the initial condition

$$S_\Lambda^{int}|_{\Lambda=\Lambda_0} = \tilde{S}^{int},$$

it determines the renormalization group flow of the effective action which corresponds to (effectively) integrating out higher momentum modes [8]. For the scalar case the initial value for the action is usually taken to be $\tilde{S}^{int} = \frac{1}{4!}\lambda \int d^D x\, \phi^4(x)$.

To see the physical meaning of Polchinski's equation it is convenient to re-write it in terms of the total effective action $S_\Lambda[\phi]$ and $\Sigma_1 = S_\Lambda - 2\hat{S}_\Lambda$. One gets

$$\Lambda \frac{\partial S_\Lambda}{\partial \Lambda} = -\frac{1}{\Lambda^2} \frac{\delta S_\Lambda}{\delta \phi} \cdot c' \cdot \frac{\delta \Sigma_1}{\delta \phi} + \frac{1}{\Lambda^2} \frac{\delta}{\delta \phi} \cdot c' \cdot \frac{\delta \Sigma_1}{\delta \phi}. \tag{4}$$

It can be checked that the kinetic term (2) is the Gaussian fixed point of this equation. Eq. (4) can also be put into the form

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda} = -\frac{1}{\Lambda^2} \frac{\delta}{\delta \phi} \cdot c' \cdot \left(\frac{\delta \Sigma_1}{\delta \phi} e^{-S_\Lambda} \right), \tag{5}$$

which shows that the partition function $\mathcal{Z} = \int \mathcal{D}\phi e^{-S_\Lambda}$ remains unchanged along the renormalization group flow. Indeed, from (5) it follows that

$$\Lambda \frac{\partial}{\partial \Lambda} \mathcal{Z} \equiv \Lambda \frac{\partial}{\partial \Lambda} \int \mathcal{D}\phi e^{-S_\Lambda} = 0.$$

A number of versions of the ERG equation for the scalar theory have been known for more than 20 years [8, 7, 9, 10, 11]. A Polchinski-type ERG equation for fermions was derived and studied in ref. [12]. In previous ERG approaches to gauge theory the authors kept the gauge fixed and allowed the effective momentum cutoff to break the gauge invariance. Then they sought to recover it in the limit when the cutoff is removed (see, for example, refs [13]). In ref. [14] one of us proposed a manifestly gauge invariant formulation of the ERG approach for pure Yang-Mills theories (see also refs [15, 16]). An important element of the formalism is an appropriate gauge invariant regularisation scheme. As it was realised later in ref. [16], it amounts to the $SU(N|N)$ super-gauge extension of the original theory, with covariant higher derivatives, but spontaneously broken in the fermionic directions, the resulting massive fields being Pauli-Villars regulating fields.

The aim of the present contribution is to give a complete formulation of this extended theory and show that it is indeed free of ultraviolet divergences and is consistent in the continuum limit *i.e.* when the cutoff $\Lambda \to \infty$. The article is organized as follows. In sec. 2 we explain the main idea of the regularisation scheme and describe in detail the structure of the $SU(N|N)$ super-gauge extension of the initial Yang-Mills theory. In sec. 3 we discuss the finiteness of the regularised theory. The full demonstration is quite long; in the present article we explain only basic points and present examples. Sec. 4 is devoted to the discussion of some potential problems arising due to the presence of a wrong sign massless vector field (loss of unitarity). We argue that in the continuum limit all the non-physical fields decouple and unitarity in the physical sector is restored. A summary of results is presented in sec. 5.

2 $SU(N|N)$ extension of $SU(N)$ Yang-Mills theory

Consider the D-dimensional pure gauge theory with the gauge group G and the action

$$S_{YM} = \frac{1}{2} \int d^D x \,\mathrm{tr}\, (F_{\mu\nu} F^{\mu\nu}). \tag{6}$$

Similar to the scalar case, as the first step of the construction of the regularised theory we modify the action by introducing the covariantised regulating function as follows:

$$\hat{S}_\Lambda = \frac{1}{2} F_{\mu\nu} \{c^{-1}\} F^{\mu\nu}. \tag{7}$$

For a given kernel W and two functions $u(x)$ and $v(x)$ this covariantisation ("wine" [14, 15]) $u\{W\}v$ is defined by

$$u\{W\}v := \mathrm{tr} \int d^D x\, u(x) W \left(-\frac{\nabla^2}{\Lambda^2} \right) \cdot v(x),$$
$$\nabla_\mu = \partial_\mu - igA_\mu,$$

where the dot means that ∇ acts via commutation.

We would like to mention that the equation for the full effective action of the manifestly gauge invariant formulation of the ERG approach of ref. [14] is

$$\Lambda\frac{\partial S_\Lambda}{\partial\Lambda} = -\frac{1}{2\Lambda^2}\frac{\delta S_\Lambda}{\delta A_\mu}\{c'\}\frac{\delta\Sigma_g}{\delta A_\mu} + \frac{1}{2\Lambda^2}\frac{\delta}{\delta A_\mu}\{c'\}\frac{\delta\Sigma_g}{\delta A_\mu},$$

where $\Sigma_g = g^2 S_\Lambda - 2\hat{S}_\Lambda$.

The $c(-\nabla^2/\Lambda^2)$ regulator implements the higher covariant derivative regularisation proposed and developed in refs [17], [18]. As it is known, however, some one-loop diagrams remain unregularised. A solution, proposed in ref. [19], is to supplement the scheme by Pauli-Villars regulating fields.

From now on we will consider the case of the gauge group $G = SU(N)$. Introducing bosonic and fermionic Pauli-Villars fields to cancel the one-loop ultraviolet divergences, one arrives at a certain system of regulating and physical fields. As discovered in ref. [16], for the purpose of regularisation it is enough and convenient to take the set of fields of the $SU(N|N)$ gauge theory. In order to generate the masses of the super-gauge fields without destroying the gauge invariance we add a scalar Higgs field. This allows masses to be generated via the Higgs mechanism, with the resulting heavy fields behaving precisely as Pauli-Villars fields. As we will explain below, the unphysical fields decouple when $\Lambda \to \infty$, *i.e.* when the regularisation is removed.

To describe the $SU(N|N)$ extension of the theory (7) with the Higgs field let us consider first the graded Lie algebra of $SU(N|M)$ (see ref. [20]). Its elements are given by Hermitian $(N+M)\times(N+M)$ matrices

$$\mathcal{H} = \begin{pmatrix} H_1 & \theta \\ \theta^\dagger & H_2 \end{pmatrix},$$

where H_N (H_M) is an $N\times N$ ($M\times M$) Hermitian matrix with complex bosonic elements, θ is an $N\times M$ matrix composed of complex Grassmann numbers, and the matrices $\mathcal{H}$ are required to be supertraceless:

$$\mathrm{str}(\mathcal{H}) := \mathrm{tr}(H_1) - \mathrm{tr}(H_2) = 0.$$

It is easy to see that the bosonic sector of the $SU(N|M)$ algebra forms the $SU(N)\times SU(M)\times U(1)$ subalgebra.

We are interested in the case when $N = M$. In this case the $2N\times 2N$ identity matrix, $1\!\!1_{2N}$, is supertraceless and, therefore, is an element of the algebra. An arbitrary element $\mathcal{H}$ of $SU(N|N)$ can be written as

$$\mathcal{H} = \mathcal{H}^0\ 1\!\!1_{2N} + \mathcal{H}^A T_A, \tag{8}$$

where T_A are the other generators of $SU(N|N)$. They can be chosen both traceless and supertraceless: $\mathrm{str}(T_A) = \mathrm{tr}(T_A) = 0$. The index A runs over $2(N^2-1)$ bosonic and $2N^2$ fermionic indices, and the Killing super-metric in the T_A subspace,

$$g_{AB} = \frac{1}{2}\mathrm{str}\,(T_A T_B)\,,$$

is symmetric when both indices A and B are bosonic, antisymmetric when both are fermionic and is zero when one is bosonic and another is fermionic.

Let us turn to the description of the $SU(N|N)$ gauge theory. The super-gauge field takes values in the graded Lie algebra of $SU(N|N)$ and, in accordance with eq. (8), can be written as

$$\mathcal{A} = \mathcal{A}^0_\mu\ 1\!\!1_{2N} + \tilde{\mathcal{A}}_\mu,$$

where

$$\tilde{\mathcal{A}}_\mu = \begin{pmatrix} A^1_\mu & B_\mu \\ \bar{B}_\mu & A^2_\mu \end{pmatrix} = \mathcal{A}^A_\mu T_A.$$

The bosonic matrix A^1_μ is the physical field, *i.e.* the field A_μ of the initial $SU(N)$ gauge theory. The fermionic field B_μ and the $SU(N)$ gauge field A^2_μ are part of the regulating structure. The pure Yang-Mills part of the action of the extended theory is equal to

$$S_{YM} = \frac{1}{2}\mathcal{F}_{\mu\nu}\{c^{-1}\}\mathcal{F}^{\mu\nu}, \tag{9}$$

where now the covariantisation is defined with the supertrace:

$$u\{W\}v := \mathrm{str}\int d^Dx\, u(x) W\left(-\frac{\nabla^2}{\Lambda^2}\right)\cdot v(x) \tag{10}$$

and

$$\begin{aligned}\nabla_\mu &= \partial_\mu - ig\mathcal{A}_\mu,\\ \mathcal{F}_{\mu\nu} &= \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig[\mathcal{A}_\mu,\mathcal{A}_\nu],\end{aligned}$$

and the regulating function c^{-1} is chosen to be a polynomial in $(-\nabla^2/\Lambda^2)$ of rank n.

To generate masses for the fermionic Pauli-Villars fields via the Higgs mechanism we add a scalar sector. The super-scalar field is given by the super-matrix

$$\mathcal{C} = \begin{pmatrix} C^1 & D \\ \bar{D} & C^2 \end{pmatrix},$$

and we take the action of the scalar sector to be

$$\mathcal{S}_{\mathcal{C}} = \nabla_\mu\cdot\mathcal{C}\{\tilde{c}^{-1}\}\nabla^\mu\cdot\mathcal{C} + \frac{\lambda}{4}\mathrm{str}\int d^Dx\,(\mathcal{C}^2-\Lambda^2)^2, \tag{11}$$

where the regulator $\tilde{c}^{-1}$ is a polynomial of rank m. The field $\mathcal{C}$ is not assumed to be supertraceless. Moreover, it acquires an expectation value which may be taken to be $<\mathcal{C}>=\Lambda\sigma_3$, where

$$\sigma_3 = \begin{pmatrix} 1\!\!1_N & 0 \\ 0 & -1\!\!1_N \end{pmatrix}.$$

Expanding the super-scalar field around the stationary point of the potential in (11), *i.e.* substituting $\mathcal{C}\to\Lambda\sigma_3+\mathcal{C}$, one obtains the following expression for the action of the scalar sector:

$$\begin{aligned}\mathcal{S}_{\mathcal{C}} &= -g^2\Lambda^2[\mathcal{A}_\mu,\sigma_3]\{\tilde{c}^{-1}\}[\mathcal{A}_\mu,\sigma_3] - 2ig\Lambda[\mathcal{A}_\mu,\sigma_3]\{\tilde{c}^{-1}\}\nabla^\mu\cdot\mathcal{C} \\ &+ \nabla_\mu\cdot\mathcal{C}\{\tilde{c}^{-1}\}\nabla^\mu\cdot\mathcal{C} + \frac{\lambda}{4}\,\mathrm{str}\int d^Dx\,\left(\Lambda\{\sigma_3,\mathcal{C}\}_+ + \mathcal{C}^2\right)^2.\end{aligned} \tag{12}$$

It is easy to see that the fields B_μ, C^1 and C^2 acquire a mass of order Λ.

Following 't Hooft's lead we choose the gauge fixing condition to be

$$\partial_\mu\mathcal{A}^\mu + ig\frac{\Lambda}{\chi}\tilde{c}^{-1}\hat{c}[\sigma_3,\mathcal{C}] = 0,$$

where $\hat{c}^{-1}(-\partial^2/\Lambda^2)$ is a polynomial of rank s. Note that it is not covariantised. The gauge fixing part of the action is equal to

$$\begin{aligned}\mathcal{S}_{GF} &= \chi\partial_\mu\mathcal{A}^\mu\cdot\hat{c}^{-1}\cdot\partial_\nu\mathcal{A}^\nu + 2ig\Lambda(\partial_\mu\mathcal{A}^\mu)\cdot\tilde{c}^{-1}\cdot[\sigma_3,\mathcal{C}] \\ &- g^2\frac{\Lambda^2}{\chi}[\sigma_3,\mathcal{C}]\cdot\tilde{c}^{-2}\hat{c}\cdot[\sigma_3,\mathcal{C}],\end{aligned} \tag{13}$$

where $u\cdot W\cdot v$ is defined as the supertrace analogue of eq. (1). One can check that the last term in the action (13) gives a mass of order Λ to the fermionic part of $\mathcal{C}$.

The Faddeev-Popov ghost super-fields are defined as

$$\eta = \begin{pmatrix} \eta^1 & \phi \\ \psi & \eta^2 \end{pmatrix},$$

where the diagonal elements are fermionic while the off-diagonal ones are bosonic. The action of the ghost sector is given by

$$\begin{aligned}\mathcal{S}_{ghost} &= -\bar{\eta}\cdot\hat{c}^{-1}\tilde{c}\cdot\partial_\mu\nabla^\mu\cdot\eta \\ &- \int d^Dx\,\mathrm{str}\left\{\frac{\Lambda}{\chi}[\sigma_3,\bar{\eta}]\left(\Lambda[\sigma_3,\eta]+[\mathcal{C},\eta]\right)\right\}.\end{aligned} \tag{14}$$

3 Finiteness of the regularised theory

In the rest of the article we consider the case $D = 4$.

The complete set of Feynman rules will be given in ref. [21]. To analyse the ultraviolet divergences in the theory we first calculate the superficial degree of divergence, $\mathcal{D}_\Gamma$, of a one-particle-irreducible diagram Γ defined in the standard way. One can easily check that it is possible to choose the numbers n, m and s, defining the ranks of the regulators, such that all diagrams with two and more loops can be made finite. Among one-loop diagrams only those with $E_{\mathcal{A}} \leq 4$, where $E_{\mathcal{A}}$ denotes the number of external $\mathcal{A}$-lines, remain unregularised by this mechanism. However, they are finite due to the cancellation of the ultraviolet divergences between the contributions of the bosonic and fermionic propagators corresponding to internal lines. This cancellation will be referred to as the supertrace mechanism.

To illustrate it, let us sketch the calculation of the one-loop diagram with two external $\mathcal{A}$-lines and two internal $\mathcal{A}$-lines. The terms of the perturbation theory expansion which generate this type of diagram, schematically omitting the Lorentz indices[1] involve the product of two vertices:

$$\mathrm{str}\left([\mathcal{A}(x), \mathcal{A}(x)]\mathcal{A}(x)\right) \mathrm{str}\left(\mathcal{A}(y)[\mathcal{A}(y), \mathcal{A}(y)]\right), \tag{15}$$

where $\mathcal{A}$ stands for the gauge field or its derivative. The leading part of the propagator between the $\mathcal{A}^A$ and $\mathcal{A}^B$ fields in the momentum representation is proportional to g^{AB}. Using the completeness relation for the generators T_A it is easy to show that by Wick pairing

$$\mathrm{str}(\mathcal{X}\mathcal{A}(x))\mathrm{str}(\mathcal{A}(y)\mathcal{Y}) = \left[\frac{1}{2}\mathrm{str}(\mathcal{X}\mathcal{Y}) - \frac{1}{4N}(\mathrm{tr}\mathcal{X}\ \mathrm{str}\mathcal{Y} + \mathrm{str}\mathcal{X}\ \mathrm{tr}\mathcal{Y})\right] \times \Delta(x-y),$$

where $\Delta(x-y)$ is a spacetime dependent factor coming from the propagator. Applying this formula to eq. (15) one can see that after the first pairing it reduces to

$$\frac{1}{2}\mathrm{str}\left([\mathcal{A}(x), \mathcal{A}(x)][\mathcal{A}(y), \mathcal{A}(y)]\right) \Delta(x-y). \tag{16}$$

Here we have used the cyclicity property of the supertrace, $\mathrm{str}(\mathcal{X}\mathcal{Y}) = \mathrm{str}(\mathcal{Y}\mathcal{X})$, which implies that $\mathrm{str}([\mathcal{X}, \mathcal{Y}]) = 0$. For the next step we use the identity

$$\mathrm{str}\left(\mathcal{X}T_A\mathcal{Y}T^A\right) = \frac{1}{2}\mathrm{str}(\mathcal{X})\mathrm{str}(\mathcal{Y}) - \frac{1}{4N}\left[\mathrm{str}(\mathcal{X}\sigma_3\mathcal{Y}) + str(\mathcal{X}\mathcal{Y}\sigma_3)\right], \tag{17}$$

valid for any super-matrices $\mathcal{X}$ and $\mathcal{Y}$, which follows from the already mentioned completeness relation. Using this identity we calculate the second $\mathcal{A}(x)$-$\mathcal{A}(y)$ pairing in eq. (16) and find that the σ_3 terms generated by (17) all cancel, as they must – to preserve the $SU(N|N)$ invariance, leaving only terms of the form $\mathrm{str}\mathcal{A}\,\mathrm{str}\mathcal{A}$ or $\mathrm{str}\mathcal{A}\,\mathrm{str}\ 1\!\!1$, both of which vanish because $\mathrm{str}\mathcal{A} = \mathrm{str}\ 1\!\!1 = 0$. This is a demonstration of the supertrace mechanism at work.

One can check by direct calculation that the supertrace mechanism ensures the finiteness of all the diagrams with two and three external $\mathcal{A}$-lines. For the diagrams with 4 external $\mathcal{A}$-lines the supertrace mechanism is not sufficient (at finite N). However, these are already finite. This follows because gauge invariant effective vertices containing less than four $\mathcal{A}$s have already been shown to be finite but gauge invariant effective vertices with a *minimum* of four $\mathcal{A}$s are already finite by power counting and Ward identities for the $SU(N|N)$ gauge theory; this derivation is given in ref. [21].

The rest of the one-loop diagrams and all the diagrams with more than one loop are finite by the higher covariant derivative regularisation. To show this we first analyse the superficial degree of divergence of the one-loop diagrams with $E_{\mathcal{A}} > 4$ and other types of external lines, as well as the two-loop vacuum diagrams. It can be shown that they are finite if and only if the ranks of the regulating functions c^{-1}, $\tilde{c}^{-1}$ and $\hat{c}^{-1}$ satisfy the inequalities

$$\begin{aligned} &s > n > m > 0, \\ &n > 2, \quad m > 1, \quad n - m > 1. \end{aligned} \tag{18}$$

[1]Beware that the commutators do not vanish once these are taken into account!

Then we show that these conditions are sufficient for the superficial degree of divergence $\mathcal{D}_\Gamma$ of the rest of the one-particle-irreducible diagrams to be

$$\mathcal{D}_\Gamma < 0.$$

This concludes the proof that all one-particle-irreducible diagrams in the $SU(N|N)$ theory with covariant higher derivatives whose action is given by eqs (9), (12), (13) and (14) are ultraviolet finite.

4 Discussion of the unphysical sector

The quadratic part of the action of the gauge sector of the $SU(N|N)$ theory is given by

$$\mathcal{S}_{YM} = \int d^D x \, \frac{1}{2} \left[\mathrm{tr}(F^1_{\mu\nu})^2 - \mathrm{tr}(F^2_{\mu\nu})^2 - 2\mathrm{tr}\,(\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu)(\partial_\mu B_\nu - \partial_\nu B_\mu) + \ldots \right] \tag{19}$$

The appearance of the term with negative sign could potentially be a problem. At first glance it seems to be a signal of an instability. However, it is rather a sign of the loss of unitarity. This can already be seen in the example of $U(1|1)$ quantum mechanics that we discuss now.

Consider the Lagrangian of a simple harmonic potential:

$$L = \frac{1}{2}\mathrm{str}\dot{\mathcal{X}}^2 - \frac{1}{2}\mathrm{str}\mathcal{X}^2,$$

where the Hermitian super-position $\mathcal{X}$ is given by the super-matrix

$$\mathcal{X} = \begin{pmatrix} x_1 & \psi \\ \bar{\psi} & x_2 \end{pmatrix}.$$

The conjugate momenta are equal to

$$p_{x_1} = \dot{x}_1, \;\; p_{x_2} = -\dot{x}_2, \;\; [x_j, p_{x_j}] = i,$$
$$p_\psi = \dot{\bar{\psi}}, \;\; p_{\bar{\psi}} = -\dot{\psi}.$$

Let us define the bosonic operators $a_j = (x_j + ip_{x_j})/\sqrt{2}$, $a_j^\dagger = (x_j - ip_{x_j})/\sqrt{2}$. They satisfy the commutation relations $[a_i, a_j] = 0$, $[a_1^\dagger, a_j^\dagger] = 0$, $[a_i, a_j^\dagger] = \delta_{ij}$. The Hamiltonian of the system is equal to

$$H = \frac{1}{2}(a_1^\dagger a_1 + a_1 a_1^\dagger) - \frac{1}{2}(a_2^\dagger a_2 + a_2 a_2^\dagger) + (\text{fermionic part}).$$

We proceed by introducing a complete set of states which includes the vacuum states $|0>_1$ and $|0>_2$ such that

$$a_1|0>_1 = 0, \quad a_2^\dagger|0>_2 = 0,$$

and n-particle states

$$|n>_1 = \frac{1}{\sqrt{n!}}(a_1^\dagger)^n|0>_1,$$
$$|n>_2 = \frac{1}{\sqrt{n!}}(a_2)^n|0>_2 \,.$$

One can see that with these definitions a_2 plays the rôle of the creation operator of the particle of the second type. With these definitions the Hamiltonian is bounded from below. In particular

$$H|n>_2 = +n|n>_2 \,.$$

Furthermore, it can be shown that it is these definitions that ensure that the vacuum preserves the $U(1|1)$ symmetry. However, the states $|n>_2$ with odd n possess negative norms:

$${}_2<n|n>_2 = \frac{1}{n!}{}_2<0|(a_2^\dagger)^n(a_2)^n|0>_2 = (-1)^n{}_2<0|0>_2.$$

This can be regarded as a violation of unitarity (negative probability). It may be mentioned that the appearance of negative norm states as a consequence of a wrong sign in part of the action is not that unusual. The Gupta-Bleuler quantization procedure relies on a modification of the Lagrangian which results in a wrong sign appearing in the A^0 part of the action.[2]

Transitions between the A^1_μ-sector and A^2_μ-sector are possible only via exchanges of fields with masses of order Λ. If Λ is finite such transitions are possible, thus leading to a violation of unitarity in the $SU(N|N)$ gauge theory. In the continuum limit, *i.e.* in the limit $\Lambda \to \infty$, all amplitudes for such transitions vanish. For example, the lowest order $A^1 A^2$ amplitude appears from the term

$$\mathrm{str}(\mathcal{A}\mathcal{A})\,\mathrm{str}(\mathcal{A}\mathcal{A}) \times (\text{UV and IR finite coefficient}). \tag{20}$$

The requirement of gauge symmetry and dimensional considerations imply that (20) is in fact proportional to

$$\sim \int d^4x\, \frac{1}{\Lambda^4} \mathrm{str}(\mathcal{F}\mathcal{F})\,\mathrm{str}(\mathcal{F}\mathcal{F})$$

and, therefore, vanishes as $\Lambda \to \infty$.

5 Summary of the results

In this article we have discussed $SU(N|N)$ gauge theory with higher covariant derivative regulators, $c\left(-\nabla^2/\Lambda^2\right)$, and the Higgs field viewed as a regularised extension of the $SU(N)$ pure Yang-Mills theory. Its structure is determined by the requirement that it can be used in the ERG equation presented in refs [14] - [16].

The extension includes the physical Yang-Mills field $A^1_\mu \equiv A_\mu$ of the initial theory and the regulating fields: the bosonic gauge field A^2_μ, the fermionic Pauli-Villars field B_μ and the scalar Pauli-Villars fields C^i. We described the field content of the $SU(N|N)$ gauge theory and wrote down its action (see eqs (9), (12), (13) and (14)). All the regulator fields except A^2_μ acquire masses proportional to the momentum cutoff Λ via the Higgs mechanism. The presence of the unphysical regulator fields lead to a source of unitarity violation in the theory with finite cutoff.

We showed that in the four-dimensional case the one-loop one-particle-irreducible diagrams with two, three or four external $\mathcal{A}$-lines are finite due to the supertrace mechanism. The rest of the one-loop one-particle-irreducible diagrams and all one-particle-irreducible diagrams with the number of loops $L \geq 2$ can be made finite by the proper choice of the regulating functions c^{-1}, $\tilde{c}^{-1}$ and $\hat{c}^{-1}$. The necessary and sufficient conditions on the parameters of these functions are given in (18).

When the regularisation is removed, *i.e.* in the limit $\Lambda \to \infty$, the massive fields B_μ and C^j become infinitely heavy and decouple. As a consequence the physical sector, which is the original $SU(N)$ Yang-Mills theory, becomes decoupled from the unphysical sector of the field A^2_μ. In this way the unitarity of the theory is restored in the continuum limit, *i.e.* when $\Lambda \to \infty$.

We expect the use of the $SU(N|N)$ regularised extension of the $SU(N)$ pure Yang-Mills theory to open up new possibilities of non-perturbative and gauge-invariant treatment of Yang-Mills theories in the framework of the ERG approach.

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[2] Of course, in the present case there is no analogue of the Gupta-Bleuler condition.

References

[1] K. Wilson and J. Kogut, Phys. Rep. **12C** (1974) 75.

[2] T.R. Morris, Phys. Lett. **B329** (1994) 241.

[3] T.R. Morris, in "New Developments in Quantum Field Theory", NATO ASI series, Plenum Press, N.Y.; `hep-th/9709100`.

[4] T.R. Morris, Prog. Theor. Phys. Suppl. **131** (1998) 395.

[5] C. Bagnuls and C. Bervillier, "Exact Renormalization Group Equations. An Introductory Review", `hep-th/0002034`.
J. Berges, N. Tetradis and C. Wetterich, "Non-Perturbative Renormalization Flow in Quantum Field Theory and Statistical Physics", `hep-ph/0005122`.

[6] D.U. Jungnickel and C. Wetterich, Prog. Theor. Phys. Suppl. **131** (1998) 495.

[7] J. Polchinski, Nucl. Phys. **B231** (1984) 269.

[8] F.J. Wegner and A. Houghton, Phys. Rev. **A8** (1973) 401.

[9] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. **B270** (1986) 687.

[10] T.R. Morris, Int. J. Mod. Phys. **A9** (1994) 2411.

[11] C. Wetterich, Nucl. Phys. **B352** (1991) 529.

[12] J. Comellas, Yu.A. Kubyshin and E. Moreno, Nucl.Phys. **B490** (1997) 653.

[13] B.J. Warr, Ann. Phys. **183** (1988) 1, 59.
C. Becchi, in "Elementary Particles, Field Theory and Statistical Mechanics", Parma, 1993; `hep-th/9607188`.
M. Bonini, M.D'Attanasio and G. Marchesini, Nucl. Phys. **B409** (1993) 441; **B418** (1994) 81; **B421** (1994) 429; **B437** (1995) 163.
U. Ellwanger, Phys.Lett. **335B** (1994) 364.
M. Reuter and C. Wetterich, Nucl.Phys. **B417** (1994) 181; **B427** (1994) 291.

[14] T.R. Morris, in "The Exact Renormalization Group", Proceedings of the Workshop. Eds. Krasnitz et al. World Scientific, Singapore, 1999, p. 1; `hep-th/9810104`.

[15] T.R. Morris, Nucl. Phys. **B573** (2000) 97; `hep-th/9910058`.

[16] T.R. Morris, JHEP **0012** (2000) 012; `hep-th/0006064`.

[17] A.A. Slavnov, Theor. Math. Phys. **13** (1972) 174.

[18] T.D. Bakeyev, A.A. Slavnov, Mod. Phys. Lett. **A11** (1996) 1539.

[19] A.A. Slavnov, Theor. Math. Phys. **33** (1977) 977.

[20] I. Bars, in "Introduction to Supersymmetry in Particle and Nuclear Physics". Eds. Castaños et al. Plenum Press, N.Y., 1984. p. 107.

[21] S. Arnone, Yu.A. Kubyshin, T.R. Morris and J.F. Tighe, SHEP preprint, in preparation.