Color-Flavor Transformation for the Special Unitary Group and Application to Low Energy QCD

J. Budczies and Y. Shnir

Abstract

The color-flavor transformation for the unitary group (Zirnbauer 1996) is extended to the special unitary group. The resulting partition function is represented as the sum over the disconnected sectors characterized by a U(1)-charge. Application to the low-energy QCD on a lattice leads to a theory where the inverse number of colors appears as an expansion parameter. We use a saddle point approximation to estimate the partition function both in the pure mesonic sector and in the case of a single baryon on a mesonic background.

1 Introduction

Effective Lagrangians have been seen to work well in the description of low energy properties of mesons and hadrons, since their introduction [1]. Initially these Lagrangians were introduced *ad hoc* by considering the symmetries of the underlying theory, QCD. Later concrete calculations schemes were given to connect the effective action with the QCD low energy lattice action, see for example [2, 3]. The purpose of this work is to introduce a new method for calculating a effective action, which rests on a recently discovered transformation [4]. The starting point is the Euclidean SU(N_c) lattice gauge theory [5] described by the action

$$S = S_{\text{gauge}} + S_{\text{quarks}}, \tag{1}$$

where S_{gauge} is the standard kinetic term for the Yang-Mills action on the lattice

$$S_{\text{gauge}} = -\frac{1}{4g^2} \sum_{\text{plaquettes}} \mathrm{Tr}(UUU^\dagger U^\dagger) + \text{c.c.} \tag{2}$$

and

$$\begin{aligned} S_{\text{quarks}} = \frac{1}{2a} \sum_{\text{sites } n} \sum_{\mu=1}^{d} & \left(\bar{q}_a^i(n)\gamma_\mu U_\mu^{\dagger ij}(n) q_a^j(n+e_\mu) - \bar{q}_a^i(n+e_\mu)\gamma_\mu U_\mu^{ij}(n) q_a^j(n)\right) \\ & + \bar{q}_a^i(n) M_{ab} q_b^i(n). \end{aligned} \tag{3}$$

is a discrete counterpart of the Dirac operator, which couples the quarks to the gauge field. Throughout this work we consider the strong coupling limit and neglect the action S_{gauge}.

In this brief report we will give only results, while proofs and details of the calculations will be presented elsewhere. The paper is organized as follows: we start by illustrating the color flavor transformation for the cases of U(N_c) and SU(N_c) (sect. 2). In sect. 3 we apply this technique to transform the partition function of strong coupling QCD and arrive at a decomposition of the partition function into disconnected sectors characterized by the baryonic charge. The integration over the quarks is performed in the pure mesonic sector and leads to a theory of a collective field Z, with N_c appearing as a factor in front of the action (sect. 4). A saddle point approximation is used to estimate the partition function in the leading order of the inverse number of colors (sect. 5). We make use of a long distance approximation in combination with a gradient expansion to recover the corresponding low energy effective action. In our approach the lattice constant a is a parameter, which has to be fixed at a certain value to yield to realistic values of the observables. In sect. 6 we give first results for the case of a single baryon on a mesonic background.

2 Color-Flavor Transformation

We start with a simple example, which illustrates the general form of the transformation [6]. For four fermion fields $\bar{q}_+, q_+$ and $\bar{q}_-, q_-$ one can easily check the relation

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} du\ \exp(\bar{q}_+ e^{iu} q_+ + \bar{q}_- e^{-iu} q_-) = 1 + \bar{q}_+ q_+ \bar{q}_- q_- = \\ &2\int_{\mathbb{C}} \frac{d\mu(z,\bar{z})}{1+z\bar{z}}\ \exp(\bar{q}_+ z \bar{q}_- - q_+ \bar{z} q_-). \end{aligned} \tag{4}$$

The integration measure $d\mu(z,\bar{z}) = \frac{\text{const.}}{(1+z\bar{z})^2}\, dz d\bar{z}$ is the natural measure of two-sphere S^2 in stereographical coordinates. In ref. [4] the following remarkable generalization of the above scheme was given:

$$\begin{aligned} &\int_{U(N_c)} dU\ \exp(\bar{q}^i_{+a} U^{ij} q^j_{+a} + \bar{q}^i_{-b} \bar{U}^{ij} q^j_{-b}) = \\ &\int_{\mathbb{C}^{N_f\times N_f}} \frac{D\mu(Z,Z^\dagger)}{\mathrm{Det}(1+ZZ^\dagger)^{N_c}}\ \exp\left(\bar{q}^i_{+a} Z_{ab} \bar{q}^i_{-b} - q^i_{+a} \bar{Z}_{ab} q^i_{-b}\right). \end{aligned} \tag{5}$$

The fermions are now vectors of N_f "flavors" $(a, b = 1, ..., N_f)$ and N_c "colors" $(i, j = 1, ..., N_c)$. The field Z becomes a $N_f \times N_f$-matrix and the integration measure

$$D\mu(Z,Z^\dagger) = \frac{\text{const.}}{\mathrm{Det}(1+ZZ^\dagger)^{2N_f}}\, dZ dZ^\dagger \tag{6}$$

is the invariant measure on the compact coset space $\mathrm{U}(2N_f)/\mathrm{U}(N_f) \times \mathrm{U}(N_f)$. For obvious reasons equation (5) is called the color-flavor transformation for the color group $\mathrm{U}(N_c)$. Note that for $N_f = 1$ the coset space is isomorphic to the two-sphere $(\mathrm{U}(2)/\mathrm{U}(1) \times \mathrm{U}(1) \cong S^2)$. Thus for $N_f = 1$ the general scheme reduces to the special case considered before.

There are color-flavor transformations for several other groups [6, 7]. Having an application to QCD in mind, we consider the special unitary group $\mathrm{SU}(N_c)$, which is the gauge group of the strong interaction $(N_c = 3)$. Again we start with the simplest case $N_c = 1$ and $N_f = 1$, where the integral on the l.h.s. of (4) reduces to a single point (evaluation at unity), and the following statement is immediate:

$$\begin{aligned} &\exp(\bar{q}_+ q_+ + \bar{q}_- q_-) = 1 + \bar{q}_+ q_+ + \bar{q}_- q_- + \bar{q}_+ q_+ \bar{q}_- q_- = \\ &2\int_{\mathbb{C}} \frac{d\mu(z,\bar{z})}{1+z\bar{z}}\ \exp(\bar{q}_+ z \bar{q}_- - q_+ \bar{z} q_-)(1 + \frac{1}{2}\bar{q}_+(1+z\bar{z})q_+ + \frac{1}{2}\bar{q}_-(1+z\bar{z})q_-). \end{aligned} \tag{7}$$

Again, cf. [8], this is a special case of a more general scheme:

$$\begin{aligned} &\int_{SU(N_c)} dU\ \exp(\bar{q}^i_{+a} U^{ij} q^j_{+a} + \bar{q}^i_{-b} \bar{U}^{ij} q^j_{-b}) = \\ &\int_{\mathbb{C}^{N_f\times N_f}} \frac{D\mu(Z,Z^\dagger)}{\mathrm{Det}(1+ZZ^\dagger)^{N_c}}\ \exp\left(\bar{q}^i_{+a} Z_{ab} \bar{q}^i_{-b} - q^i_{+a} \bar{Z}_{ab} q^i_{-b}\right) \sum_{Q=-N_f}^{N_f} \chi_Q(Z, Z^\dagger, q, \bar{q}). \end{aligned} \tag{8}$$

The physical interpretation of equation (8) is as follows: It is a sum over contributions, which come from mesonic excitations over the vacuum $(Q = 0)$ or mesonic excitations in the presence of $|Q|$ baryons $(Q > 0)$ or $|Q|$ antibaryons $(Q < 0)$. The explicit expressions for χ_Q in the vacuum sector and in the one-baryon sector are [8]

$$\chi_0 = \text{const.}, \quad \chi_1 = \text{const.} \sum_{\sigma\in S_{N_c}} \mathrm{sgn}\,\sigma \prod_{i=1}^{N_c} \bar{q}^i_{+a}(1+ZZ^\dagger)_{ab} \bar{q}^{\sigma(i)}_{+b}. \tag{9}$$

The sum runs over all permutations S_{N_c} of the numbers $1, ..., N_c$.

3 Lattice Gauge Theory

We consider a Euclidean SU(N_c) gauge theory in d dimensions on a hypercubic lattice with lattice constant a. We restrict our considerations to an even number d of spacetime dimensions. The fermions $\psi(n)$ are placed on the lattice sites labeled by n, while the gauge field $U_\mu(n)$ is put on the lattice links, which may be labeled by a lattice site n and a direction $\mu = 1, .., d$. The gauge theory is described by the partition function

$$\mathcal{Z}(q,\bar{q}) = \prod_{n,\mu} \int_{\mathrm{SU}(N_c)} dU_\mu(n)\, \exp(-S_{n,\mu}(U_\mu(n)))\,, \tag{10}$$

where the fermions on two neighboring sites are coupled through the gauge field on the connecting link in the gauge invariant way

$$\begin{aligned} S_{n,\mu}(U_\mu(n)) =& \frac{1}{2a}\left(\bar{q}^i_a(n)\gamma_\mu U^{\dagger ij}_\mu(n) q^j_a(n+e_\mu) - \bar{q}^i_a(n+e_\mu)\gamma_\mu U^{ij}_\mu(n) q^j_a(n)\right) \\ &+ \bar{q}^i_a(n) M_{ab} q^i_b(n). \end{aligned} \tag{11}$$

The action (11) is a discrete version of the gauge covariant Dirac operator. Here we do not worry about the fermion doubling problem and employ naive fermions. The mass matrix $M = \mathrm{diag}(m_1, ..., m_{N_f})$ contains the quark masses and is diagonal in flavor space. The Dirac spinor fields carry spacetime indices, which take values $\nu = 1, ..., N_s$ with $N_s = 2^{d/2}$, and are coupled through the γ-matrices. The γ-matrices are the generators of a Clifford algebra over the Euclidean spacetime and satisfy the anticommutation relations $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$. Now we apply the color-flavor transformation to the gauge field $U_\mu(n)$ separately on each link. In doing so, we get a new action, which is a function of the flavor field $Z_\mu(n)$, which is again situated at the links of the lattice. The result is a sum over contributions from sectors with different baryonic charge Q and can be written as

$$\mathcal{Z}(q,\bar{q}) = \prod_{n,\mu} \int_{\mathbb{C}^{N_f\times N_f}} D\mu_{N_c}(Z_\mu(n), Z^\dagger_\mu(n)) \sum_{Q=-N_f}^{N_f} \chi_{Q,\mu}(n) \exp\left(-S_{n,\mu}(Z_{n,\mu})\right) \tag{12}$$

with the integration measure (6). The color-flavor transformed action for a link (n,μ) reads

$$\begin{aligned} S_{n,\mu}(Z_\mu(n)) =& \frac{1}{2a}\Big(\bar{q}^i_a(n+e_\mu)\gamma_\mu Z_{\mu,ab}(n) q^i_b(n+e_\mu) + \bar{q}^i_a(n)\gamma_\mu Z^\dagger_{\mu,ab}(n) q^i_b(n)\Big) \\ &+ \bar{q}^i_a(n) M_{ab} q^i_b(n). \end{aligned} \tag{13}$$

Explicit expressions for the prefactors in the case of the vacuum sector ($Q = 0$) and the one-baryon sector ($Q = 1$) are

$$\chi_{0,\mu}(n) = \text{const.}, \tag{14}$$

$$\chi_{1,\mu}(n) = \text{const.} \sum_{\sigma\in S_{N_c}} \mathrm{sgn}\,\sigma \prod_{i=1}^{N_c} \bar{q}^i_a(n+e_\mu)(1 + Z_\mu Z^\dagger_\mu)_{ab}(n) \bar{q}^{\sigma(i)}_b(n). \tag{15}$$

4 Integration over the Quarks and Saddle Point

Performing the Gaussian integral over the quarks in the pure mesonic sector ($Q = 0$) and sending the result back to the exponent, we obtain the action

$$\begin{aligned} S_{Q=0}(Z) = N_c\Bigg(&-\sum_n \mathrm{Tr}\ln\left(\sum_{\mu=1}^{d}\gamma_\mu\left(Z_\mu(n-e_\mu) + Z^\dagger_\mu(n)\right) - aM\right) \\ &+ \sum_n \sum_{\mu=1}^{d} \mathrm{Tr}\ln\left(1 + Z_\mu(n) Z^\dagger_\mu(n)\right)\Bigg). \end{aligned} \tag{16}$$

Variation with respect to the independent variables Z and $Z^\dagger$ yields the saddle point equations

$$\begin{aligned}\gamma_\mu\left(\frac{1}{Z_\mu(n)}+Z_\mu^\dagger(n)\right) &= \sum_{\nu=1}^{d}\gamma_\nu\left(Z_\nu(n-e_\nu)+Z_\nu^\dagger(n)\right)-2aM,\\ \gamma_\mu\left(\frac{1}{Z_\mu^\dagger(n-e_\mu)}+Z_\mu(n-e_\mu)\right) &= \sum_{\nu=1}^{d}\gamma_\nu\left(Z_\nu(n-e_\nu)+Z_\nu^\dagger(n)\right)-2aM.\end{aligned} \tag{17}$$

The solution of the saddle point equations is

$$Z_\mu^0(n)=Z_\mu^{0\dagger}(n)=z\gamma_\mu I \qquad \text{with } z=\frac{1}{2d-1}\left(aM\pm\sqrt{2d-1+(aM)^2}\right), \tag{18}$$

where I is the unit matrix in flavor space. Recall the following definitions of γ_5 and the projectors on the chiral components of the spinors

$$\gamma_5 = i^{d(d-1)/2}\gamma_1\cdot\ldots\cdot\gamma_d, \tag{19}$$

$$\gamma_L=\frac{1}{2}(1+\gamma_5),\quad \gamma_R=\frac{1}{2}(1-\gamma_5). \tag{20}$$

In the chiral limit ($M=0$) there is a saddle point manifold

$$Z_\mu^0(n)=Z_\mu^{0\dagger}(n)=z\gamma_\mu g^{\gamma_5}:=z\gamma_\mu(\gamma_L\otimes g+\gamma_R\otimes g^{-1}), \tag{21}$$

which is parameterized by unitary matrices $g\in \mathrm{U}(N_f)$. The lattice gauge theory in the chiral limit ($M=0$) is invariant under the chiral transformations ($(g_L,g_R)\in \mathrm{U}(N_f)_L\times \mathrm{U}(N_f)_R$) of the quarks fields

$$\begin{aligned} q(n)&\to(\gamma_L\otimes g_L+\gamma_R\otimes g_R)\,q(n),\\ \bar q(n)&\to\left(\gamma_L\otimes g_R^{-1}+\gamma_R\otimes g_L^{-1}\right)\bar q(n).\end{aligned} \tag{22}$$

The chiral symmetry of the action is preserved by the color-flavor transformation and leads to the invariance of the action (16) and the saddle point manifold under the transformations

$$\begin{aligned} Z_\mu(n)&\to\left(\gamma_L\otimes g_L^{-1}+\gamma_R\otimes g_R^{-1}\right)Z_\mu(n)\left(\gamma_L\otimes g_L+\gamma_R\otimes g_R\right),\\ Z_\mu^\dagger(n)&\to\left(\gamma_L\otimes g_L^{-1}+\gamma_R\otimes g_R^{-1}\right)Z_\mu^\dagger(n)\left(\gamma_L\otimes g_L+\gamma_R\otimes g_R\right).\end{aligned} \tag{23}$$

The saddle point (18) is invariant under (23) only if $g_L=g_R$, i.e. it breaks the the chiral $\mathrm{U}(N_f)_L\times\mathrm{U}(N_f)_R$ symmetry to the subgroup $\mathrm{U}(N_f)$. The above symmetry considerations explain how chiral symmetry breaking takes place in our approach to the gauge theory. The chiral symmetry breaking gives rise to Goldstone bosons, which can be identified with the coset space $\mathrm{U}(N_f)_L\times\mathrm{U}(N_f)_R/\mathrm{U}(N_f)\cong\mathrm{U}(N_f)$ and parameterize the saddle point manifold (21).

5 Gradient Expansion

Our aim is to derive an effective theory, which describes the long range behavior of Goldstone modes $g\in\mathrm{U}(N_f)$. We use the technique of a long distance approximation in combination with a gradient expansion around the saddle point manifold, as it was developed in [9, 10]. The lattice action (16) is replaced by a simple continuum action

$$S_{Q=0}(Z)\longrightarrow S(g)=S_{\text{fl}}(g)+S_M(g), \tag{24}$$

where $S_{\text{fl}}(g)$ is the action corresponding to fluctuations of the Goldstone modes, and $S_M(g)$ the contribution due to finite quark masses. More explicitly, $Z_\mu(n)$ is put into correspondence with a continuous field $g(x)\in\mathrm{U}(N_f)$ in the following way:

$$Z_\mu(n)=Z_\mu^\dagger(n)=z\gamma_\mu\, g(n+\frac{a}{2}e_\mu)^{\gamma_5}. \tag{25}$$

Inserting into (16) the Taylor expansion

$$g(n+\frac{a}{2}e_\mu) = g(n) + \frac{a}{2}\partial_\mu g(n) + \frac{1}{2!}\left(\frac{a}{2}\right)^2 \partial_\mu^2 g(n) + \cdots, \tag{26}$$

we obtain for the hypercubic lattice

$$S_{\mathrm{fl}}(g) = \frac{N_s}{8d}a^{2-d}\int d^dx \ \mathrm{Tr}\left(\partial g \partial g^{-1}\right), \tag{27}$$

$$S_M(g) = N_s\frac{\sqrt{2d-1}}{2d}a^{-d}\int d^dx \ \mathrm{Tr}\left(aM(g+g^{-1})\right). \tag{28}$$

We have carried out the expansion up to fourth order derivatives. In particular, we have done so for a body-centered hypercubic (BHC) lattice in $d=4$ spacetime dimensions. The 4-dimensional BHC lattice has the advantage of leading to a continuum action with complete rotational O(4) symmetry. The results will be presented elsewhere [11].

In $d=4$ spacetime dimensions both parts of the actions (27) and (28) diverge when the lattice constant approaches zero. Recall that we are considering the strong coupling limit only, i.e. we neglect the kinetic term (2), which is suppressed as $1/g^2$. Therefore our theory is restricted to the low energy sector and we have to keep the lattice constant at a finite value. There are three ways to estimate the lattice constant a through comparisons with experimental data: Using the relation

$$<\bar{q}_f q_f> = -\frac{1}{V}\frac{\partial}{\partial m_f}\bigg|_0 S_{\mathrm{saddle}} = -N_sN_c\frac{d-1}{d\sqrt{2d-1}}a^{1-d}, \tag{29}$$

we get from the experimental value for the chiral condensate $a=(166\ \mathrm{MeV})^{-1}$. By looking at the coefficient in front of the fluctuation action,

$$\frac{F_\pi^2}{4} = N_sN_c\frac{1}{8d}\,a^{2-d}, \tag{30}$$

we get from the experimental value for the pion decay constant F_π the estimate $a=(76\ \mathrm{MeV})^{-1}$. By looking at the coefficient in front of the fluctuation action,

$$\frac{1}{4}F_\pi^2m_\pi^2 = N_sN_c\frac{\sqrt{2d-1}}{2d}\,a^{1-d}m_f, \tag{31}$$

we get from the experimental values for the pion decay constant, the mass of the pion m_π and the mass of the light quarks m_f the value $a=(76\ \mathrm{MeV})^{-1}$. We conclude that the lattice constant has to be chosen as $a\approx(100\ \mathrm{MeV})^{-1}\approx 2$ Fermi to get a realistic description.

6 Static Baryon

We consider a single static baryon on a mesonic background. In doing so we deal with the $Q=1$ sector right on the worldline of the baryon and the $Q=0$ sector away from the baryon. Performing the integration over the quarks we have to take into account factors in front of the exponential along the worldline of the baryon, which stem from the expression for χ_1 (15). We introduce periodic boundary conditions in the time direction. Introducing the quantity

$$M(n) := \sum_{\mu=1}^{d}\gamma_\mu\left(Z_\mu(n-e_\mu)+Z_\mu^\dagger(n)\right) - aM \tag{32}$$

we are able to express the action for the static baryon on the mesonic background by means of the following propagator along the wordline

$$\mathsf{G} := \prod_{n\in\ \mathrm{worldline}} G(n), \quad \text{where } G(n) := \left(1+Z_1(n)Z_1^\dagger(n)\right)M(n)^{-1}\gamma_1. \tag{33}$$

The result is

$$S_{Q=1} = S_{Q=0} - \frac{1}{N_c} \ln \sum_{\sigma \in \hat{S}_{N_c}} \mathcal{N}(\sigma) \prod_{l=1}^{N_c} \left(\mathrm{Tr}\,\mathsf{G}^l\right)^{c_l(\sigma)} + \text{const.}, \tag{34}$$

where the sum extends over conjugation classes of permutations. The class of a permutation is determined by the lengths of its cycles. We denote the number of cycles of length l of the permutation σ by $c_l(\sigma)$. The normalization constants in (34) are given by $\mathcal{N}(\sigma) = \left(\prod_{l=1}^{N_c} l^{c_l(\sigma)} c_l(\sigma)!\right)^{-1}$. For the lowest numbers of colors we get the explicit expressions

$$\begin{aligned} &\mathrm{Tr}\,\mathsf{G} \quad (N_c = 1), \quad (\mathrm{Tr}\,\mathsf{G})^2 + \mathrm{Tr}\,\mathsf{G}^2 \quad (N_c = 2), \\ &(\mathrm{Tr}\,\mathsf{G})^3 + 3\,\mathrm{Tr}\,\mathsf{G}^2\,\mathrm{Tr}\,\mathsf{G} + 2\,\mathrm{Tr}\,\mathsf{G}^3 \quad (N_c = 3). \end{aligned} \tag{35}$$

The next step would be to obtain the saddle point equations for the static baryon. In [8] we study a solution of the saddle point equations in $d = 2$ spacetime dimensions.

7 Summary and Acknowledgments

We have briefly described a new approach to lattice QCD, which is useful in the low energy domain, where the quarks are confined to mesons and baryons. In principle models away from the strong coupling limit can be treated with the color flavor transformation, but one has to replace the standard action (2) by an action that can be generated with help of an auxiliary field [4, 12].

We are indebted to M. R. Zirnbauer for suggesting the subject to us and we thank him for fruitful discussions. We thank S. Nonnenmacher for collaborating with on the implementation of the SU(N_c) color-flavor transformation and its application to the static baryon. Useful discussions with A. Altland are acknowledged.

References

[1] S. Weinberg, Physica **96A** (1979) 327.

[2] N. Kawamoto and J. Smit, Nucl. Phys. **B192** (1981) 100.

[3] S. Myint and C. Rebbi, Nucl. Phys. **B421** (1994) 241, hep-lat/9401009.

[4] M. R. Zirnbauer, J. Phys. **A29** (1996) 7113, chao-dyn/9609007.

[5] K. Wilson, Phys. Rep. **C12** (1976) 331.

[6] M. R. Zirnbauer, in *Proceedings of the XIIth International Congress of Mathematical Physics (Brisbane, July 1997)* (1998), chao-dyn/9810016.

[7] G. Beuchelt, *Dirac Fermions in 2+1 Dimensions with Random Mass Distribution*, Master's thesis, Köln (1997).

[8] J. Budczies, S. Nonnenmacher and Y. Shnir, to be published.

[9] A. Altland and B. D. Simons, J.Phys. **A32** (1999) L353, cond-mat/9811134.

[10] A. Altland and B. D. Simons, Nucl. Phys. **B562** (1999) 445, cond-mat/9909152.

[11] J. Budczies and Y. Shnir, to be published.

[12] J. Budczies and M. R. Zirnbauer, in preparation.