The Gegenbauer Polynomial Technique: evaluation of complicated Feynman integrals

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Abstract

We discuss a progress in calculation of Feynman integrals due to the use of the Gegenbauer Polynomial Technique and demonstrate the results for most complicated parts of $O(1/N^3)$ contributions to critical exponents of ϕ^4-theory for any space-time dimensionality D.

1 Basic Formulae

The method based on the expansion of propagators in Gegenbauer series (see [1]) has been introduced in [2, 3] fifteen years ago. One has shown [2, 4] that by this method the analytical evaluation of counterterms in the minimal subtraction scheme at the 4-loop level in any model and for any composite operator was indeed possible. The Gegenbauer Polynomial (GP) technique has been applied successfully to the propagator-type Feynman diagrams (FD) in many calculations (see [2, 3]). In the present paper we consider *a development of the GP technique* (obtained in [5]) and some application of the results [6].

Throughout the paper we use the following notations. The use of dimensional regularization is assumed. All the calculations are performed in the space of dimension $D = 4 - 2\varepsilon$. Note that contrary to [2], we analyze FD directly in momentum x-space which allows us to avoid the appearance of Bessel functions. Because we consider here only propagator-type massless FD, we know their dependence on a single external momentum beforehand. The point of interest is the coefficient function C_f which depends on $D = 4 - 2\varepsilon$ and is a Laurent series in ε.

1. First of all, we present some useful formulas for the Gegenbauer polynomials. Following [2, 3], D-space integration can be represented in the form

$$d^Dx \; = \; \frac{1}{2} S_{D-1} (x^2)^{\lambda} dx^2 d\hat{x} \qquad (\lambda = D/2-1),$$

where $\hat{x} = \vec{x}/\sqrt{x^2}$, and $S_{D-1} = 2\pi^{\lambda+1}/\Gamma(\lambda+1)$ is the surface of the unit hypersphere in R_D. The Gegenbauer polynomials $C_n^{\delta}(t)$ are defined as [1, 3]

$$(1-2rt+r^2)^{-\delta} \; = \; \sum_{n=0}^{\infty} C_n^{\delta}(t) r^n \quad (r \leq 1), \qquad C_n^{\delta}(1) \; = \; \frac{\Gamma(n+2\delta)}{n!\Gamma(2\delta)}, \tag{1}$$

whence the expansion for the propagator is:

$$\frac{1}{(x_1-x_2)^{2\delta}} \; = \; \sum_{n=0}^{\infty} C_n^{\delta}(\hat{x_1}\hat{x_2}) \left[\frac{(x_1^2)^{n/2}}{(x_2^2)^{n/2+\delta}} \Theta(x_2^2 - x_1^2) \; + \; \left(x_1^2 \longleftrightarrow x_2^2 \right) \right], \tag{2}$$

where

$$\Theta(y) \; = \; \begin{cases} 1, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases}$$

Orthogonality of the Gegenbauer polynomials $C_n^{\lambda}(x)$ is expressed by the equation (see [2])

$$\int C_n^{\lambda}(\hat{x_1}\hat{x_2}) \; C_m^{\lambda}(\hat{x_2}\hat{x_3}) \, d\hat{x_2} \; = \; \frac{\lambda}{n+\lambda} \, \delta_n^m \; C_n^{\lambda}(\hat{x_1}\hat{x_3}), \tag{3}$$

where δ_n^m is the Kronecker symbol.

The following formulas are useful (see [2, 7]):

$$\begin{aligned} C_n^{\delta}(x) &= \sum_{p\geq 0} \frac{(2x)^{n-2p}(-1)^p\Gamma(n-p+\delta)}{(n-2p)!p!\Gamma(\delta)} \qquad \text{and} \\ \frac{(2x)^n}{n!} &= \sum_{p\geq 0} C_{n-2p}^{\delta}(x)\frac{(n-2p+\delta)\Gamma(\delta)}{p!\Gamma(n-p+\delta+1)} \end{aligned} \tag{4}$$

Substituting the latter equation from (4) for $\delta = \lambda$ in the first one, we have the following equation after the separate analyses at odd and even n:

$$C_n^{\delta}(x) = \sum_{k=0}^{[n/2]} C_{n-2p}^{\lambda}(x)\frac{(n-2k+\lambda)\Gamma(\lambda)}{k!\Gamma(\delta)}\frac{\Gamma(n+\delta-k)\Gamma(k+\delta-\lambda)}{\Gamma(n+\lambda+1-k)\Gamma(\delta-\lambda)} \tag{5}$$

2. Following [2, 7] we introduce the traceless product (TP) $x^{\mu_1\cdots\mu_n}$ connected with the usual product $x^{\mu_1}...x^{\mu_n}$ by the following equations

$$\begin{aligned} x^{\mu_1\cdots\mu_n} &= \hat{S}\sum_{p\geq 0}\frac{n!(-1)^p\Gamma(n-p+\lambda)}{2^{2p}p!(n-2p)!\Gamma(n+\lambda)}\, g^{\mu_1\mu_2}...g^{\mu_{2p-1}\mu_{2p}}\; x^{2p}\; x^{\mu_{2p+1}}...x^{\mu_n}. \\ x^{\mu_1}...x^{\mu_n} &= \hat{S}\sum_{p\geq 0}\frac{n!\Gamma(n-2p+\lambda+1)}{(2)^{2p}p!(n-2p)!\Gamma(n-p+\lambda+1)}\, g^{\mu_1\mu_2}...g^{\mu_{2p-1}\mu_{2p}}\; x^{2p}\; x^{\mu_{2p+1}\cdots\mu_n}. \end{aligned} \tag{6}$$

Comparing Eqs.(4) and (6), we obtain the following relations between TP and GP

$$z^{\mu_1\cdots\mu_n}\; x^{\mu_1\cdots\mu_n} = \frac{n!\Gamma(\lambda)}{2^n\Gamma(n+\lambda)}\; C_n^{\lambda}(\hat{x}\hat{z})\; \left(x^2z^2\right)^{n/2}, \quad x^{\mu_1\cdots\mu_n}\; x^{\mu_1\cdots\mu_n} = \frac{\Gamma(n+2\lambda)\Gamma(\lambda)}{2^n\Gamma(2\lambda)\Gamma(n+\lambda)}\; x^{2n} \tag{7}$$

We give also the simple, but useful relations:

$$z^{\mu_1\cdots\mu_n}\; x^{\mu_1\cdots\mu_n} = z^{\mu_1}...\,z^{\mu_n}\; x^{\mu_1\cdots\mu_n} = z^{\mu_1\cdots\mu_n} x^{\mu_1}...\,x^{\mu_n}, \tag{8}$$

which follow immediately from the TP definition: $g^{\mu_i\mu_j}\; x^{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_n} = 0$.

The use of the TP $x^{\mu_1\cdots\mu_n}$ makes it possible to ignore the terms of the type $g^{\mu_i\mu_j}$ that arise upon integration: they can be readily recovered from the general structure of the TP. Therefore, in the process of integration, it is only necessary to keep the coefficient of the leading term $x^{\mu_1}...x^{\mu_n}$. The rules of integration of a FD containing TP can be found, for example, in [7, 8, 9]. For a loop we have (hereafter $Dx \equiv (d^Dx)/(2\pi)^D$) [1]:

$$\int Dx\frac{x^{\mu_1\cdots\mu_n}}{x^{2\alpha}(x-y)^{2\beta}} = \frac{1}{(4\pi)^{D/2}}\;\frac{y^{\mu_1\cdots\mu_n}}{y^{2(\alpha+\beta-\lambda-1)}}\; A^{n,0}(\alpha,\beta), \tag{9}$$

where

$$A^{n,m}(\alpha,\beta) = \frac{a_n(\alpha)a_m(\beta)}{a_{n+m}(\alpha+\beta-\lambda-1)} \quad \text{and} \quad a_n(\alpha) = \frac{\Gamma(D/2-\alpha+n)}{\Gamma(\alpha)}$$

Note that in our analysis it is necessary to consider more complicated cases of integration, when the integrand contains Θ functions. Indeed, using the Eqs.(2) and (7), we can represent the propagator $(x_1-x_2)^{-2\lambda}$ in the following form [2]:

$$\frac{1}{(x_1-x_2)^{2\lambda}} = \sum_{n=0}^{\infty}\frac{2^n\Gamma(n+\lambda)}{n!\Gamma(\lambda)}\; x_1^{\mu_1\cdots\mu_n}\; x_2^{\mu_1\cdots\mu_n}\left[\frac{1}{x_2^{2(\lambda+n)}}\Theta(x_2^2-x_1^2) + \left(x_1^2\longleftrightarrow x_2^2\right)\right] \tag{10}$$

[1] Eq.(9) has been used in [7, 8, 9, 10] for calculations of the moments of structure functions of deep inelastic scattering.

[2] In the case of the propagator $(x_1-x_2)^{-2\delta}$ with $\delta\neq\lambda$ we should also use Eq.(5).

Using the GP properties from previous subsection and the connection (7) between GP and TP, we obtain the rules for calculating FD with the Θ-terms and TP.

3. *The rules* have the following form:

$$\int Dx \frac{x^{\mu_1\cdots\mu_n}}{x^{2\alpha}(x-y)^{2\beta}}\;\Theta(x^2-y^2) \;=\; \frac{1}{(4\pi)^{D/2}}\,\frac{y^{\mu_1\cdots\mu_n}}{y^{2(\alpha+\beta-\lambda-1)}}\sum_{m=0}^{\infty}\frac{B(m,n|\beta,\lambda)}{m+\alpha+\beta-\lambda-1}$$
$$\stackrel{(\beta=\lambda)}{=}\; \frac{1}{(4\pi)^{D/2}}\,\frac{y^{\mu_1\cdots\mu_n}}{y^{2(\alpha-1)}}\,\frac{1}{\Gamma(\lambda)}\,\frac{1}{(\alpha-1)(n+\lambda)} \qquad (11)$$

$$\int Dx \frac{x^{\mu_1\cdots\mu_n}}{x^{2\alpha}(x-y)^{2\beta}}\;\Theta(y^2-x^2) \;=\; \frac{1}{(4\pi)^{D/2}}\,\frac{y^{\mu_1\cdots\mu_n}}{y^{2(\alpha+\beta-\lambda-1)}}\sum_{m=0}^{\infty}\frac{B(m,n|\beta,\lambda)}{m+n-\alpha+\lambda+1}$$
$$\stackrel{(\beta=\lambda)}{=}\; \frac{1}{(4\pi)^{D/2}}\,\frac{y^{\mu_1\cdots\mu_n}}{y^{2(\alpha-1)}}\,\frac{1}{\Gamma(\lambda)}\,\frac{1}{(n+\lambda+1-\alpha)(n+\lambda)} \qquad (12)$$

where

$$B(m,n|\beta,\lambda) \;=\; \frac{\Gamma(m+\beta+n)}{m!\,\Gamma(m+n+1+\lambda)\Gamma(\beta)}\;\frac{\Gamma(m+\beta-\lambda)}{\Gamma(\beta-\lambda)}$$

The sum of above diagrams does not contain Θ-terms and should reproduce Eq.(9). To compare the r.h.s. of Eqs.(11,12) and the r.h.s. of Eq.(9) we use the transformation of ${}_3F_2$-hypergeometric function with unit argument ${}_3F_2(a,b,c;e,b+1;1)$ (see [11]):

$$\sum_{k=0}^{\infty}\frac{\Gamma(k+a)\Gamma(k+c)}{k!\,\Gamma(k+f)}\,\frac{1}{k+b} \;=\; \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(c-b)}{\Gamma(f-b)\Gamma(1+b-a)}$$
$$-\;\frac{\Gamma(1-a)\Gamma(a)}{\Gamma(f-c)\Gamma(1+c-f)}\sum_{k=0}^{\infty}\frac{\Gamma(k+c-f+1)\Gamma(k+c)}{k!\,\Gamma(k+1+c-a)}\,\frac{1}{k+c-b} \qquad (13)$$

In this case (when $k=m, b=\alpha+\beta-\lambda-1, c=n+\beta$) we compare Eq.(9) with the sum of Eqs.(11,12).

Analogously to Eqs.(11) and (12), we have more complicate cases:

$$\int Dx \frac{x^{\mu_1\cdots\mu_n}}{x^{2\alpha}(x-y)^{2\beta}}\Theta(x^2-z^2) \;=\; \frac{1}{(4\pi)^{D/2}}\;y^{\mu_1\cdots\mu_n}\Bigg[\frac{\Theta(y^2-z^2)}{y^{2(\alpha+\beta-\lambda-1)}}\;A^{n,0}(\alpha,\beta)$$
$$+\;\sum_{m=0}^{\infty}\frac{B(m,n|\beta,\lambda)}{z^{2(\alpha+\beta-\lambda-1)}}\left(\left(\frac{y^2}{z^2}\right)^{m}\frac{\Theta(z^2-y^2)}{m+\alpha+\beta-\lambda-1}\;-\;\left(\frac{z^2}{y^2}\right)^{m+\beta+n}\frac{\Theta(y^2-z^2)}{m-\alpha+n+1+\lambda}\right)\Bigg]$$
$$\stackrel{(\beta=\lambda)}{=}\;\frac{1}{(4\pi)^{D/2}}\,\frac{1}{\Gamma(\lambda)}\;y^{\mu_1\cdots\mu_n}\Bigg[\frac{1}{y^{2(\alpha-1)}}\frac{\Theta(y^2-z^2)}{(\alpha-1)(n+\lambda+1-\alpha)}$$
$$+\;\frac{1}{z^{2(\alpha-1)}}\frac{1}{n+\lambda}\left(\frac{\Theta(z^2-y^2)}{\alpha-1}\;-\;\left(\frac{z^2}{y^2}\right)^{n+\lambda}\frac{\Theta(y^2-z^2)}{n+1+\lambda-\alpha}\right)\Bigg] \qquad (14)$$

$$\int Dx \frac{x^{\mu_1\cdots\mu_n}}{x^{2\alpha}(x-y)^{2\beta}}\Theta(z^2-y^2) = \frac{1}{(4\pi)^{D/2}}\;y^{\mu_1\cdots\mu_n}\Bigg[\frac{\Theta(z^2-y^2)}{y^{2(\alpha+\beta-\lambda-1)}}\;A^{n,0}(\alpha,\beta)$$
$$-\;\sum_{m=0}^{\infty}\frac{B(m,n|\beta,\lambda)}{z^{2(\alpha+\beta-\lambda-1)}}\left(\left(\frac{y^2}{z^2}\right)^{m}\frac{\Theta(z^2-y^2)}{m+\alpha+\beta-\lambda-1}\;-\;\left(\frac{z^2}{y^2}\right)^{m+\beta+n}\frac{\Theta(y^2-z^2)}{m-\alpha+n+1+\lambda}\right)\Bigg]$$
$$\stackrel{(\beta=\lambda)}{=}\;\frac{1}{(4\pi)^{D/2}}\,\frac{1}{\Gamma(\lambda)}\;y^{\mu_1\cdots\mu_n}\Bigg[\frac{1}{y^{2(\alpha-1)}}\frac{\Theta(z^2-y^2)}{(\alpha-1)(n+\lambda+1-\alpha)}$$
$$-\;\frac{1}{z^{2(\alpha-1)}}\frac{1}{n+\lambda}\left(\frac{\Theta(z^2-y^2)}{\alpha-1}\;-\;\left(\frac{z^2}{y^2}\right)^{n+\lambda}\frac{\Theta(y^2-z^2)}{n+1+\lambda-\alpha}\right)\Bigg] \qquad (15)$$

One can easily see that the sum of the above diagrams leads to results identical to (9).

2 Calculation of complicated FD

The aim of this section is to demonstrate the result of [5] for a class of master two-loop diagrams containing the vertex with two propagators having index 1 or λ.

Let us consider the following general diagram

$$\int \frac{DxDy}{y^{2\alpha}(z-y)^{2t}(z-x)^{2\beta}x^{2\gamma}(x-y)^{2s}} \equiv J(\alpha,t,\beta,\gamma,s)$$

and restrict ourselves to the FD $A(\alpha,\beta,\gamma) = J(\alpha,\lambda,\beta,\gamma,\lambda)$ which is the one of FD of interest for us here. It is easily shown (see [12, 9, 5]) that ($\sigma = 3+\lambda-(\alpha+\beta+\gamma)$)

$$C_f[A(\alpha,\beta,\gamma)] = C_f[A(\alpha,\sigma,\gamma)] = C_f[J(\gamma,\lambda,\lambda,\sigma,\alpha)] = C_f[J(\sigma,\gamma,\lambda,\lambda,\beta)], \tag{16}$$

Making the Fourier transformation of both the diagram $A(\alpha,\beta,\gamma)$ and its solution in the form $C_f\left[A(\alpha,\beta,\gamma)\right](z^2)^{-\tilde{\sigma}}$, where hereafter $\tilde{t} = \lambda+1-t$, $t = \{\alpha,\beta,\gamma,\sigma,...\}$ and considering the new diagram as one in the momentum x-space, we obtain the relation

$$\begin{aligned} C_f[A(\alpha,\beta,\gamma)] &= \frac{a_0^2(\lambda)a_0(\alpha)a_0(\beta)a_0(\gamma)}{a_0(\delta)}\, C_f\left[J(\tilde{\alpha},1,\tilde{\beta},\tilde{\gamma},1)\right] \\ &= a_0^2(\lambda)a_0(\alpha)a_0(\beta)a_0(\gamma)a_0(\sigma)\; C_f\left[J(\tilde{\beta},1,\tilde{\alpha},\tilde{\gamma},1)\right] \end{aligned} \tag{17}$$

between the diagram which contains the vertex with two propagators having the index λ and the similar diagram containing the vertex with two propagators having the index 1.

Repeating manipulations of (see [12, 9, 5]), we obtain the following relations:

$$C_f[J(\tilde{\beta},1,\tilde{\alpha},\tilde{\gamma},1)] = C_f[J(\tilde{\beta},1,\tilde{\sigma},\tilde{\gamma},1)] = C_f[J(1,1,\tilde{\gamma},\tilde{\sigma},\tilde{\alpha})] = C_f\left[J(\tilde{\sigma},1,1,\tilde{\gamma},\tilde{\beta})\right] \tag{18}$$

Thus, we have obtained the relations between all diagrams from the class introduced in the beginning of this section. Hence, it is necessary to find the solution for one of them. We prefer to analyze the diagram $A(\alpha,\beta,\gamma)$, that is the content of the next subsection.

1. *We calculate the diagram* $A(\alpha,\beta,\gamma)$ in the following way[3]:

$$\begin{aligned} A(\alpha,\beta,\gamma) &\overset{(10)}{=} \sum_{n=0}^{\infty} \frac{2^n\Gamma(n+\lambda)}{n!\Gamma(\lambda)} \int DxDy \frac{z^{\mu_1\cdots\mu_n}}{x^{2\gamma}(z-x)^{2\beta}} \frac{y^{\mu_1\cdots\mu_n}}{y^{2\alpha}(x-y)^{2\lambda}} \Big[\frac{\Theta(z^2-y^2)}{z^{2(n+\lambda)}} + \frac{\Theta(y^2-z^2)}{y^{2(n+\lambda)}}\Big] \\ &\overset{(14,15)}{=} \frac{1}{(4\pi)^{D/2}} \frac{1}{\Gamma(\lambda)} \frac{1}{\alpha-1} \sum_{n=0}^{\infty} \frac{2^n\Gamma(n+\lambda)}{n!\Gamma(\lambda)} \int Dx \frac{z^{\mu_1\cdots\mu_n}x^{\mu_1\cdots\mu_n}}{x^{2\gamma}(z-x)^{2\beta}} \cdot \Bigg[\frac{1}{\lambda+n+1-\alpha} \\ &\times \Big(\frac{\Theta(z^2-x^2)}{z^{2(n+\lambda)}x^{2(\alpha-1)}} + \frac{\Theta(x^2-z^2)}{x^{2(n+\lambda)}z^{2(\alpha-1)}}\Big) - \frac{1}{\lambda+n+\alpha-1} \cdot \Big(\frac{\Theta(z^2-x^2)}{z^{2(n+\lambda+\alpha-1)}} + \frac{\Theta(x^2-z^2)}{x^{2(n+\lambda+\alpha-1)}}\Big)\Bigg] \end{aligned} \tag{19}$$

After some algebra, we get (see [5]) the result in the form:

$$C_f[A(\alpha,\beta,\gamma)] = \frac{1}{(4\pi)^D} \frac{1}{\Gamma(\lambda)} \frac{1}{\alpha-1} \Big[\overline{I} - \tilde{I}\Big],$$

where [4]

[3] The symbol $\overset{(n)}{=}$ marks the fact that the equation (n) is used at this step.

[4] We would like to note that the coefficients in Eqs.(20) and (21) are similar to ones (see [13]) which appeared in calculations of FD with massive propagators having the mass m. Representation of the results for these diagrams in the form $\sum \varphi_n(z^2/m^2)^n$ (φ_n are the coefficients which are similar to those in Eqs.(20) and (21)) is very convenient to obtain the results for more complicated FD by integration over m of the results of a less complicated FD (see [14, 15]).

$$
\begin{aligned}
\overline{I} &= \sum_{n=0}^{\infty} \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)} \left[\frac{1}{\lambda+n+1-\alpha} \cdot \Big(A^{n,0}(\alpha-1+\gamma,\beta) + A^{n,0}(n+\lambda+\gamma,\beta) \Big) \right. \\
&- \left. \frac{1}{\lambda+n+\alpha-1} \cdot \Big(A^{n,0}(\gamma,\beta) + A^{n,0}(n+\alpha+\lambda+\gamma-1,\beta) \Big) \right] \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I} &= \frac{\Gamma(1-\beta)\Gamma(\lambda+1-\alpha)\Gamma(\lambda-1+\alpha)\Gamma(1-\beta+\lambda)\Gamma(1-\gamma)\Gamma(\alpha+\beta+\gamma-\lambda-2)}{\Gamma(2\lambda)\Gamma(2+\lambda-\alpha-\beta)\Gamma(\alpha+\gamma-1)\Gamma(2+\lambda-\gamma-\beta)\Gamma(\alpha+\beta-\lambda-1)} \\
&- \sum_{n=0}^{\infty} \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)} \frac{(-1)^n\Gamma(1-\beta)}{\Gamma(\beta-\lambda)} \cdot \frac{1}{\lambda+n+\alpha-1} \qquad (21) \\
&\times \left[\frac{\Gamma(\alpha+\beta+\gamma-\lambda-2)\Gamma(2-\alpha-\gamma)}{\Gamma(3-\alpha-\beta-\gamma-n)\Gamma(\alpha+\gamma+\lambda-1+n)} + \frac{\Gamma(1-\gamma)\Gamma(\beta+\gamma-\lambda-1)}{\Gamma(\gamma-\lambda-n)\Gamma(2+2\lambda-\beta-\gamma+n)} \right]
\end{aligned}
$$

Thus, a simple solution for $A(\alpha,\beta,\gamma)$ is obtained[5]. In the next section, we shall consider the important special case of these results.

2. *As a simple, but important example* to apply these results, we consider the diagram $J(1,1,1,1,\alpha)$. It arises in the framework of a number of calculations (see [12, 17, 18, 19, 20]). Its coefficient function $I(\alpha) \equiv C_f[J(1,1,1,1,\alpha)]$ can be found (see [5]) as follows

$$
I(\alpha) = \frac{a_0^4(1)a_0(\alpha)}{a_0(\alpha+2-2\lambda)}\, C_f[J(\lambda,\lambda,\lambda,\lambda,\tilde{\alpha})] \quad \text{and} \quad C_f[J(\lambda,\lambda,\lambda,\lambda,\tilde{\alpha})] = C_f[A(\tilde{\alpha},3-\lambda-\tilde{\alpha},\lambda)]
$$

From Eqs. (20) and (21) we obtain

$$
\begin{aligned}
I(\alpha) &= -\frac{2}{(4\pi)^D} \frac{\Gamma^2(\lambda)\Gamma(\lambda-\alpha)\Gamma(\alpha+1-2\lambda)}{\Gamma(2\lambda)\Gamma(3\lambda-\alpha-1)} \qquad (22) \\
&\times \left[\frac{\Gamma^2(1/2)\Gamma(3\lambda-\alpha-1)\Gamma(2\lambda-\alpha)\Gamma(\alpha+1-2\lambda)}{\Gamma(\lambda)\Gamma(2\lambda+1/2-\alpha)\Gamma(1/2-2\lambda+\alpha)} + \sum_{n=0}^{\infty} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\alpha+1)} \frac{1}{n+1-\lambda+\alpha} \right]
\end{aligned}
$$

Note that in [17] Kazakov has got another result for $I(\alpha)$:

$$
\begin{aligned}
I(\alpha) &= -\frac{2}{(4\pi)^D} \frac{\Gamma^2(\lambda)\Gamma(1-\lambda)\Gamma(\lambda-\alpha)\Gamma(\alpha+1-2\lambda)}{\Gamma(2\lambda)\Gamma(\alpha)\Gamma(3\lambda-\alpha-1)} \qquad (23) \\
&\times \left[\frac{\Gamma(\lambda)\Gamma(2-\lambda)\Gamma(\alpha)\Gamma(3\lambda-\alpha-1)}{\Gamma(2\lambda-1)\Gamma(3-2\lambda)} - \sum_{n=0}^{\infty} (-)^n \frac{\Gamma(n+2\lambda)}{\Gamma(n+2-\lambda)} \left(\frac{1}{n+1-\lambda+\alpha} + \frac{1}{n+2\lambda-\alpha} \right) \right]
\end{aligned}
$$

From Eqs. (22) and (23) we obtain the transformation rule for ${}_3F_2$-hypergeometric function with argument -1:

$$
\begin{aligned}
{}_3F_2(2a,b,1;b+1,2-a;-1) &= b \cdot \frac{\Gamma(2-a)\Gamma(b+a-1)\Gamma(b-a)\Gamma(1+a-b)}{\Gamma(2a)\Gamma(1+b-2a)} \qquad (24) \\
&- \frac{1-a}{b+a-1} \cdot {}_3F_2(2a,b,1;b+1,b+a;1) - \frac{b}{1+a-b} \cdot {}_3F_2(2a,1+a-b,1;2+a-b,2-a;-1),
\end{aligned}
$$

where $a=\lambda$ and $b=1-\lambda+\alpha$ are used.

Equation (24) has been explicitly checked for $a=1$ and $b=2-a$ (i.e. $\lambda=1$ and $\alpha=1$), where the ${}_3F_2$-hypergeometric functions may be calculated exactly. It is very difficult to prove Eq.(24) for arbitrary values of a and b; the general proof seems to be non-trivial. Note that it is different from the equations of [11, 21] and may be considered as a new transformation rule.

[5] Possibility to represent $C_f[A(\alpha,\beta,\gamma)]$ as a combination of ${}_3F_2$-hypergeometric functions with a unit argument, has been observed before in [16].

3 Applications

The above results have been used for evaluation of very complicated FD which appear in calculations in various types of $1/N$ expansions:

- In the calculation (in [22]) of the next-to-leading (NLO) corrections to the value of dynamical mass generation (see [23]) in the framework of three-dimensional Quantum Electrodynamics.
- In the evaluation (in [24]) of the correct value of of the leading-order contribution to the β-function of the θ-term in Chern-Simons theory. The β-function is zero in the framework of usual perturbation theory, but it has nonzero values in $1/N$ expansion (see [25]).
- In the evaluation (in [20]) of NLO corrections to the gluon Regge trajectory (see discussions in [20] and references therein).
- In the calculation (in [26]) of the next-to-leading corrections to the BFKL intercept of spin-dependent part of high-energy asymptotics of hadron-hadron cross-sections.
- In the calculation (in [26]) of the next-to-leading corrections to the BFKL equation at arbitrary conformal spin.
- In the evaluation (in [6]) of the most complicated parts of $\mathrm{O}(1/N^3)$ contributions to critical exponents of ϕ^4-theory, for any spacetime dimensionality D.

We consider here only basic steps of the last analysis [6]. Since the pioneering work of the St Petersburg group [12, 27], exploiting conformal invariance [28] of critical phenomena, it was known that the $\mathrm{O}(1/N^3)$ terms in the large-N critical exponents of the non-linear σ-model, or equivalently ϕ^4-theory, in any number D of space-time dimensions, are of maximal complexity in the Feynman integral $I(\lambda)$ (see [27]):

$$I(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\Delta} \ln \Pi(\lambda, \Delta) \Bigg|_{\Delta=0}, \tag{25}$$

where

$$\Pi(\lambda, \Delta) = \frac{x^{2(\lambda+\Delta)}}{\pi^D} \int\int \frac{\mathrm{d}^D y \mathrm{d}^D z}{y^2 z^2 (x-y)^{2\lambda} (x-z)^{2\lambda} (y-z)^{2(\lambda+\Delta)}} \tag{26}$$

is a two-loop two-point integral, with three dressed propagators, made dimensionless by the appropriate power of x^2.

The result, obtained by GP technique, is

$$I(\lambda) = \Psi(1) - \Psi(1-\lambda) + \frac{\Phi(\lambda) - \frac{1}{3}\Psi''(\lambda) - \frac{7}{24}\Psi''(1)}{\Psi'(1) - \Psi'(\lambda)}, \tag{27}$$

where $\Psi(x) = \Gamma'(x)/\Gamma(x)$ and

$$\Phi(\lambda) = 4\int_0^1 dx \frac{x^{2\lambda-1}}{1-x^2} \left\{ \mathrm{Li}_2(-x) - \mathrm{Li}_2(-1) \right\} \quad (\mathrm{Li}_2(x) = \sum_{n>0} x^n/n^2), \tag{28}$$

In [29, 6] the integral $I(\lambda)$ has been expanded near $D = 2$ and $D = 3$, respectively, (i.e. for $D = 2 - 2\varepsilon$ and $D = 3 - 2\varepsilon$) up to ε^8 in the form of alternative and non-alternative double Euler sums [30, 31].

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