Restriction on the Slope Parameter of the Isgur–Wise Function

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Abstract

Integral representation for Isgur – Wise function (IWF) is obtained in the framework of instant–form relativistic Hamiltonian dynamics for mesons with one heavy quark. The upper and lower limits on the slope parameter of IWF are calculated in the model-independent way. IWF is calculated for different wave functions of quarks in the meson. The difference between the limits on the slope parameter equals 1/3. The constraint on the slope parameter is in a good agreement with experiments. Weak dependence of IWF on the choice of wave functions is found.

Composite quark systems with one heavy quark have been the focus of attention in the last several years. In the infinite heavy quark mass limit, the additional spin-flavour symmetries appear and simplify calculation of matrix elements of electroweak transitions [1, 2]. Therefore, all meson form factors of the electroweak transitions are determined by a single universal Isgur – Wise function (IWF) $\xi_{IW}(w)$ [1] ($w = (v \cdot v')$, v'_μ and v_μ are 4–velocities of initial and final meson).

Calculation of this function is nonperturbative in principle. Several approaches exist to this problem, for example, the calculations on lattices [3], QCD sum rules [2], [4] – [9], Nambu-Jona-Lasinio model [10], quasipotential approach [11, 12], and different forms of relativistic Hamiltonian dynamics (RHD): light-front dynamics [13, 14], point [15] and instant forms [16, 17].

For the study of the properties of IWF the slope parameter $\rho^2 = -\xi'_{IW}(w = 1)$ is of special interest. This interest, in particular, is explained by its recent measurements [18] – [20]. By these experiments, one can discriminate between different models and, additionally, constraint parameters of the models. The results of calculation of the slope parameter in different approaches are very different [2] – [4], [6], [10, 11, 13, 14, 16, 17].

In this work, IWF is calculated for mesons which contain one heavy quark. A strong restriction is obtained for the slope parameter ρ^2. This result depends weakly on a model of interaction of quarks. The interval of restriction on the slope parameter equals 1/3.

The calculations of IWF are performed for different phenomenological wave functions of quarks in meson. It is shown that IWF depends weakly enough on the choice of the model wave functions (about 10% at $w \leq 2$).

In our work we use the instant form of RHD in the version developed by the authors [21, 22](RHD is discussed in greater detail in Ref. [23] and references therein). Our approach is closely related to that in Refs. [16, 17]. However, our approach contains some original features, e.g., the method of construction of current transition matrix elements [22].

In our work, IWF is calculated from the semileptonic decay of pseudoscalar meson form factors [22]. The hadron part of invariant amplitude of the semileptonic decay of pseudoscalar meson can be expressed through two form factors $F_\pm$:

$$\langle \vec{p}_c | J^\mu | \vec{p}\,'_{c'} \rangle = P^\mu_- \; F_-(t) + P^\mu_+ \; F_+(t) \; , \tag{1}$$

where $\vec{p}_c$, $\vec{p}\,'_{c'}$ are 3-momenta of mesons, $P^\mu_- = (p_c - p'_{c'})^\mu$, $P^\mu_+ = (p_c + p'_{c'})^\mu$, $t = (p_c - p'_{c'})^2$.

In our approach, the form factors $F_-(t)$, $F_+(t)$ have the following integral representation in the impulse approximation [22]:

$$F_\pm(t) = \int_{m_Q + m_q}^{\infty} d\sqrt{s}\, d\sqrt{s'} \varphi_c(k(s))\, G^{(0)}_\pm(s,t,s')\, \varphi_{c'}(k(s')) \; , \tag{2}$$

where $G^{(0)}_\pm(s, Q^2, s')$ are the so-called free two-partical form factors, $\varphi_{c'}$, φ_c are phenomenological wave functions of initial and final mesons in the sense of RHD. They are normalized with relativistic density states

$$\varphi(k) = \sqrt{\sqrt{s}(1 - (m_Q^2 - m_q^2)^2/s^2)}\; u(k)\, k \; , \quad \int k^2\, u^2(k)\, d\, k \; = \; 1 \; , \tag{3}$$

where $k(s) = \sqrt{(s^2 - 2s(m_Q^2 + m_q^2) + (m_Q^2 - m_q^2)^2)/4\,s}$, m_Q , m_q are masses of heavy and light quarks respectively, $u(k)$ is the non-relativistic wave function.

Free two-partical form factors $G_\pm^{(0)}$ have been calculated in Ref. [22].

The matrix element of the semileptonic decay of pseudoscalar meson can be determined in terms of form factors $h_\pm(w)$ (see, for example, [13]):

$$\langle \vec{p}_c | J^\mu | \vec{p}\,'_{c'} \rangle = \sqrt{M_c M_{c'}}\,[h_+(w)(v+v')^\mu + h_-(w)(v-v')^\mu] \ . \tag{4}$$

Here $M_{c'}, M_c$ are masses of mesons in initial and final states.

The form factors in Eq. (4) are connected to the form factors in Eqs. (1), (2) by the following formula:

$$h_\pm(w) = \frac{1}{2\sqrt{M_c M'_{c'}}}\,[F_+(t)\,(M_c \pm M'_{c'}) + F_-(t)\,(M_c \mp M'_{c'})] \ . \tag{5}$$

Scalar product of 4–velocities is determined through the square of momentum transfer:
$w = (M_c^2 + M_{c'}^2 - t)/(2M_c M_{c'})$. The IWF can be obtained from expressions (2), (5) by finding the limit $M_2 \sim M_3 \sim m_Q \to \infty$, $M_1 = m_q$:

$$\lim_{m_Q \to \infty} h_+(w) = \xi_{IW}(w) \ , \qquad \lim_{m_Q \to \infty} h_-(w) = 0 \ . \tag{6}$$

In our calculations, we have assumed that the wave functions in Eq. (2) become independent on the flavour of heavy quark and coincide in initial and final states in limit (6).

The result of calculation of IWF can be presented in following form:

$$\xi_{IW}(w) = \frac{1}{\sqrt{2(w+1)}} \int_0^\infty \int_{-1}^1 k^2\,dk\,dz \sqrt[4]{\frac{k'^2 + m_q^2}{k^2 + m_q^2}} \cos\omega\, u(k)\, u(k') \ , \tag{7}$$

$$k' = \left[(\sqrt{w^2-1}kz + w\sqrt{k^2+m_q^2})^2 - m_q^2 \right]^{1/2} \ ,$$

$$\omega = \arctan \frac{k\sqrt{(w^2-1)(1-z^2)}}{(w+1)m_q + \sqrt{k^2+m_q^2} + \sqrt{k'^2+m_q^2}} \ ,$$

The condition $\xi_{IW}(1) = 1$ is a consequence of the normalization (3).

In analogy with the works [15] – [17], the slope parameter of IWF at $w = 1$ can be represented as the sum of three terms:

$$\rho^2 = \rho^2_{space} + \rho^2_{WR} + \rho^2_{quark} \ , \tag{8}$$

$$\rho^2_{space} = \frac{1}{3}\int_0^\infty k^2\,(k^2+m_q^2)\left(\frac{d\,u}{d\,k}\right)^2 d\,k + \frac{1}{6}\int_0^\infty \frac{k^4\,u^2(k)}{k^2+m_q^2}\,d\,k - \frac{1}{2} \ , \tag{9}$$

$$\rho^2_{WR} = \frac{1}{6}\int_0^\infty \frac{k^4\,u^2(k)}{(\sqrt{k^2+m_q^2}+m_q)^2}\,d\,k \ , \qquad \rho^2_{quark} = \frac{1}{4} \ .$$

The terms in Eq. (8) describe the contributions of the wave function (with relativistic normalization (3)), Wigner rotation, and quark current, respectively. The contribution ρ^2_{WR} and the second term in ρ^2_{space} have relativistic nature. Let us remark that our value of ρ^2_{quark} coincides with that in [16].

The integrals of the wave function can be transformed in the space coordinate representation:

$$\int_0^\infty k^2 \left(\frac{d\,u}{d\,k}\right)^2 d\,k = \int_0^\infty r^4\,\psi_0^2(r)\,d\,r = \langle r^2 \rangle \ ,$$

$$\int_0^\infty k^4 \left(\frac{d\,u}{d\,k}\right)^2 d\,k = \int_0^\infty r^4 \left(\frac{d\,\psi_0}{d\,r}\right)^2 d\,r \ , \quad \int_0^\infty r^2\,\psi_0^2(r)\,d\,r = 1. \tag{10}$$

Here $\langle r^2 \rangle$ is the mean square radius in the ground state with wave function $\psi_0(r)$.

Note that all integrals in Eqs. (8), (9) are positive. From equations (8), (9), (10), the restriction on possible values of ρ^2 can be established. To do so, let us consider the relativistic terms in Eqs. (8), (9) (ρ^2_{WR} and the second term in ρ^2_{space}).

In the non-relativistic limit, these positive terms disappear in Eq. (9), and we obtain the lower bound for the slope parameter:

$$\rho^2 \geq \frac{1}{3}\int_0^\infty r^4 \left(\frac{d\,\psi_0}{d\,r}\right)^2 d\,r + \frac{1}{3}m_q^2\left\langle r^2\right\rangle - \frac{1}{4}\,, \tag{11}$$

In the ultrarelativistic limit these relativistic terms in Eq. (8) take the maximum values and we obtain the upper bound for ρ^2:

$$\rho^2 \leq \frac{1}{3}\int_0^\infty r^4 \left(\frac{d\,\psi_0}{d\,r}\right)^2 d\,r + \frac{1}{3}m_q^2\left\langle r^2\right\rangle + \frac{1}{12}\,. \tag{12}$$

Let us remark that the difference between these maximum and minimum of the slope parameter equals 1/3.

Now we shall use the results of works [24]. In these works a recurrence formula was derived:

$$\left\langle r^{2l}\right\rangle = \frac{1}{2\,\mu}l(2l+1)\frac{\left\langle r^{2(l-1)}\right\rangle}{E_1(l)-E_0}\,. \tag{13}$$

Here l is the orbital angular momentum, $\left\langle r^{2l}\right\rangle$ multipole moments in ground state of two-body system, E_0 is the ground state energy, $E_1(l)$ is the energy of the lowest level of multipolarity l, μ reduced mass, in the limit (6), $\mu = m_q$. Recurrent formula (13) is satisfied within 1% accuracy as a relation for confining potential.

For estimate of mean square radius in (11) and (12) by means of Eq. (13), we have used two measurements of energy levels of D - meson ($1S_0$ and $1P_1$) [25].

For further estimate, let us transform the integrals in expressions (11) and (12):

$$\frac{1}{3}\int_0^\infty r^4 \left(\frac{d\,\psi_0}{d\,r}\right)^2 d\,r = 1 + \frac{1}{3}\int_0^\infty \psi_0(r)\left[(\vec{r}\,\hat{\vec{p}})^2 - i\,(\vec{r}\,\hat{\vec{p}})\right]\psi_0(r)\,r^2\,d\,r\,. \tag{14}$$

Here $\hat{\vec{p}}$ is the operator of canonical 3-momentum of the system.

For a model-independent estimate of the integral in right hand part of Eq. (14), we have used quasiclassical approximation. In ground state, this approximation gives $(\vec{r}\,\vec{p}) = 0$ because the mean value of $r(t)$ sweeps a circle in the quasiclassical approximation. Therefore, the left hand side of Eq. (14) is approximately equal to unity.

To estimate quantum corrections to this result, it is possible to calculate exactly the integral in the left part of Eq. (14) in some models. Let us perform the calculation for the model with linear confinement and Coulomb behavior at small distances [26]:

$$u(r) = N_T\, e^{-\alpha r^{3/2}-\beta r}\,,\ \alpha = \frac{2}{3}\sqrt{2\,\mu\,a}\,,\ \beta = \mu\,b\,. \tag{15}$$

In Eq.(15), a and b are parameters of linear and Coulomb parts of potential, respectively. We use the value of $b = 4/3\alpha_s$. In doing so, we have used the value $\alpha_s = 0.52$ at a scale of heavy meson masses. The parameter of linear part of potential $a = 0.0816$ GeV^2 was obtained by fitting the mean square radius in ground state of D–meson which was calculated from Eq. (13). In this way, the quantum correction $\simeq$0.07 was obtained.

As follows from the general conditions of application of the quasiclassical approximation, the less steep potential ($\sim r^\alpha$, $\alpha < 1$ or $\sim \ln r$) must result in smaller quantum corrections. Hence, for such class of models, the following restriction is valid to within $\simeq$0.07 accuracy:

$$\frac{3}{4} \leq \rho^2 - \frac{1}{3}\,m_q^2\,\left\langle r^2\right\rangle \leq \frac{13}{12}\,. \tag{16}$$

The light quark mass m_q is now the only free parameter in our estimations . In our relativistic calculation in Refs. [21], [22], we used the value $m_q = 0.25$ GeV. This calculation are in reasonably good agreement

with experiments on electroweak properties of light mesons. In this case, Eq. (13) gives $\langle r^2 \rangle = 0.422$ fm^2. Hence, inequalities (16) are reduced to the following bound:

$$0.98 \leq \rho^2 \leq 1.31 \tag{17}$$

If the mass m_q is varied within the limits in which the modern relativistic calculations are performed in different works ($0.20 \leq m_q \leq 0.33$ GeV), then the region of possible ρ^2 increases:

$$0.93 \leq \rho^2 \leq 1.38 \tag{18}$$

The derived restrictions are examined in our work for the following different model wave functions: the harmonic-oscillator potential model (see, for example, [27])

$$u(k) = N_{HO} \exp\left(-k^2/2\,b^2\right) , \tag{19}$$

the power law wave function (see, for example, [28])

$$u(k) = N_{PL}\,(k^2/b^2 + 1)^{-n} , \quad n = 2, 3 , \tag{20}$$

and already mentioned wave function with linear confinement and a Coulomb behavior at small distances which is described by Eq. (15).

i) The wave function parameters can be determined by fitting the mean square radius of the $D-$ meson which was calculated by formula (13). The fitting gives the following values: in the model corresponding to Eq. (19) we obtain $b = 0.372$ GeV, in the model (20) we find $b = 0.526$ GeV for n =2 and $b = 0.731$ GeV for n =3 . With the wave functions presented above, the following values for slope parameter in Eq. (8) were obtained: $\rho^2 = 1.32$ in the model (19), $\rho^2 = 1.13$ in the model (20) for n =2 and $\rho^2 = 1.20$ for n =3, and $\rho^2 = 1.20$ in the model (15).

In the model of harmonic oscillator, the result satisfies inequalities (17) within the accuracy of our calculations. The calculations with exact formulas (8), (9) for models (15), (20) satisfy the restriction (17).

ii) The wave function parameters can be determined as well by fitting the lepton decay constant of the $B-$ meson $f_B = 0.180$ GeV which was calculated in work [29]. The fitting gives the following values: in the model corresponding to Eq. (19) we obtain $b = 0.5471$ GeV, in the model (20), we find $b = 0.5606$ GeV for n =2 and $b = 0.9179$ GeV for n =3, in the model (15), we obtain $a = 0.2158$ GeV^2 by fixed $\alpha_s = 0.52$. With the presented wave functions, the following values for the slope parameter in Eq. (8) were obtained: $\rho^2 = 1.27$ in the model (19), $\rho^2 = 1.12$ in the model (20) for n =2 and $\rho^2 = 1.15$ for n =3, and $\rho^2 = 1.15$ in the model (15). The calculations with exact formulas (8), (9) for models (15), (19), (20) satisfy the restriction (17).

Let us compare our results (17), (18) with recent experimental values of ρ^2. The result of CLEO Collaboration [18], [20] ($\rho^2 = 1.67 \pm 0.11(\text{stat}) \pm 0.22(\text{syst})$) satisfies our "soft" result (18) and is slightly outside the region (17). The average LEP result [19], [20] ($\rho^2 = 1.13 \pm 0.08(\text{stat}) \pm 0.16(\text{syst})$) satisfies the "hard" restriction (17).

Let us consider results for ρ^2 in other approaches. Two restrictions on the slope parameter ρ^2 were obtained in QCD sum rules: $\rho^2 \geq 0.25$ [4] , $\rho^2 \leq 1.7$ [9]. Quasipotential approach gives $\rho^2 \geq 0.81$ [12]. Another restriction was obtained in the framework of RHD [16] $\rho^2 \geq 3/4$. As we see, our restriction (17) does not contradict these results but reinforces them. However, our result, as well as that of work [16] and experiments [18], [19], is in poor correlation with the result of Ref. [5] obtained in the framework of QCD sum rules $\rho^2 \leq 0.75 \pm 0.15$.

Now let us consider calculations in the framework of instant form of RHD. In the work [17], calculations are performed for different interaction models of quarks in meson and results are within the range from 0.97 to 1.28, that is very close to (17). The result of quasipotential approach [11] coincides with ours, $\rho^2 = 1.02$.

The results of different calculations of ρ^2 in the framework of QCD sum rules differ considerably: 1.00 ± 0.02 [8] , 0.84 ± 0.02 [6] , 0.66 ± 0.05 [2] , 0.70 ± 0.25 [7] . The first and fourth of these values are near our range (17).

The results of the Nambu-Jona-Lasinio model [10], $\rho^2 = 1.25$, are in good correlation with our result.

The lattice calculations [3] are in good correlation with our result, but we should point the large errors of these calculations: $\rho^2 = 0.9^{+0.2+0.4}_{-0.3-0.2}$.

Now let us discuss briefly calculation of IWF in our approach. In fig. 1 and fig. 2, the results of calculation for IWF with the described model wave functions are shown. Parameters of wave functions in 1 were determined by fitting of MSR of D–meson (point i)). Parameters of wave functions in 2 were determined by fitting the B–meson lepton decay constant (point ii)). The calculation was carried out at m_q=0.25 GeV.

As can be seen from the fig. 1 and fig. 2, one of the most interesting features of our calculations is the weak dependence of IWF on the choice of the model wave functions at $w \leq 2$. Hence, the properties of IWF are mainly determined by the relativistic kinematics of light quark or so-called "light" degrees of freedom.

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References

[1] N. Isgur and M.B. Wise, Phys. Lett. B232 (1989) 113; B237 (1990) 527.

[2] M. Neubert, Phys. Rev. D45 (1992) 2451.; D47 (1993) 4063.

[3] K.C. Bowler *et al.* UKQCD Collab., Phys.Rev. D52 (1995) 5067.

[4] J.D. Bjorken, Preprint of Stanford Univ., SLAC-PUB-5278 (1990).

[5] M.B. Voloshin, Phys. Rev. D46 (1992) 3062.

[6] E. Bagan , P. Ball and P. Gosdzinsky, Phys. Lett. B301 (1993) 101.

[7] B. Blok, M. Shifman, Phys. Rev. D47 (1993) 2949.

[8] M. Narison, Phys.Lett. B325 (1994) 197.

[9] E. de Rafael and J. Taron, Phys. Rev. D50 (1994) 373.

[10] D.Ebert, T.Feldmann, R.Friedrich and H.Reinhardt, Nucl. Phys. B434 (1995) 619; D. Ebert, T. Feldmann and H. Reinhardt, Phys. Lett. B388 (1996) 154; T. Feldmann, arXiv hep-ph/9606451

[11] R.N. Faustov and V.O. Galkin, Z. Phys. C66 (1995) 119.

[12] D. Ebert, R.N. Faustov, V.O. Galkin, Phys.Rev.D, 61 (2000) 014016.

[13] H. Cheng, C. Cheung and C. Hwang, Phys. Rev. D55 (1996) 1559.

[14] S. Simula, Phys.Lett. B373 (1996) 193.

[15] B.D. Keister, arXiv hep-ph/9703310

[16] A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Phys. Lett. B365 (1995) 319; Phys. Lett. B386 (1996) 304.

[17] V. Morénas, A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Phys. Lett. B386 (1996) 315; Phys. Rev. D56 (1997) 5668.

[18] J.P.Alexander *et al.*(CLEO Collab.) Preprint CLEO CONF 00–03, 2000.

[19] D. Abbaneo *et al.* (LEP Collaborations) Preprint CERN–EP–2000–016, 2000.

[20] M.Battaglia, arXiv:hep–ex/0009026.

[21] E.V. Balandina, A.F. Krutov and V.E. Troitsky, Theor.Math.Fiz. 103 (1995) 41; J.Phys. G22 (1996) 1585.

[22] E.V. Balandina, A.F. Krutov, V.E. Troitsky and O.I. Shro, Yad.Fiz. 63 (2000) 301; A.F. Krutov, V.E. Troitsky and O.I. Shro, *Proceedings of XIIth Workshop on QFTHEP*, p.275. ed. B.B.Levtchenko, Moscow State University, 1999.

[23] B.D. Keister and W.N. Polyzou, Advances in Nucl. Phys. 20 (1991) 225.

[24] R.J. Lombard and J. Mareš, Phys. Rev. D59 (1999) 076005; Phys. Lett. B472 (2000) 150.

[25] D.E. Groom et al., Europ. Phys. J. C15 (2000) 1.

[26] H. Tezuka, J. Phys.A: Math.Gen. 24 (1991) 5267.

[27] P.L. Chung, F. Coester and W.N. Polyzou, Phys. Lett. B205 (1988) 545.

[28] F. Schlumpf, Phys. Rev. D50 (1994) 6895.

[29] B decays, revised 2nd edition, ed. Stone S., World Sci., 1993.

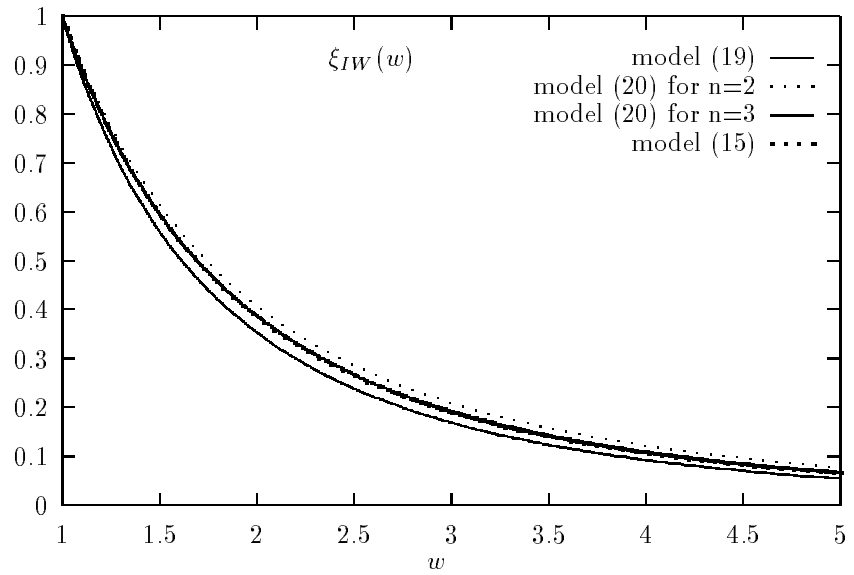

Figure 1: Results of the calculations of Isgur–Wise function with the different model wave functions. Wave function parameters were determined by fitting of mean square radius of $D-$ meson (see point i) in text).

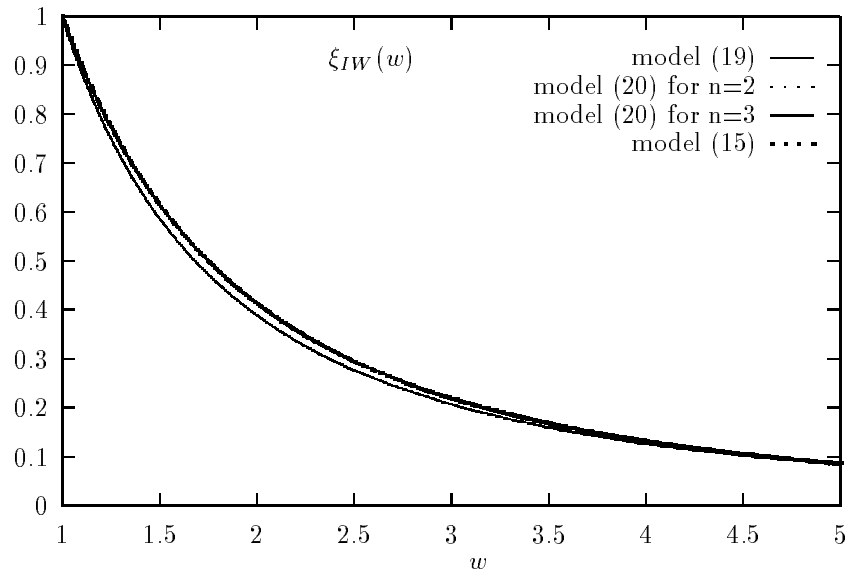

Figure 2: Results of the calculations of Isgur–Wise function with the different model wave functions. Wave function parameters were determined by fitting of lepton decay constant of $B-$ meson (see point ii) in text).