Regular Representations of the Generalized Heisenberg Algebra

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Abstract

General properties of regular irreducible representations of generalized Heisenberg algebra in any weakly completed, nondegenerate inner product space are investigated.

The Heisenberg algebra, i.e. the algebra of canonical commutation relations was studied intensively for a long time (see review [1]). The representations of this algebra were constructed not only in Hilbert space, but in an indefinite metric space as well [2,3]. In the last years q- and R-deformations of the Heisenberg algebra were investigated in many papers (see [4 - 6] and references therein).

All these algebras are special cases of the algebra, which we call the generalized Heisenberg algebra. Some properties of this algebra were considered in [5, 6] in connection with properties of q- and R-deformed Heisenberg algebra.

The present paper considers general properties of this algebra.

In case of one degree of freedom we say that two operators a and its adjoint a^+ generate generalized Heisenberg algebra , if there exists a self-adjoint operator N which satisfies the conditions:

$$[N, a] = -a, \qquad [N, a^+] = a^+. \tag{1}$$

We consider conditions (1) as equivalent to the following ones:

$$N\, a = a\,(N-1), \qquad N\, a^+ = a^+\,(N+1). \tag{1'}$$

a and a^+ are adjoint under, in general, indefinite inner product $< \cdot, \cdot >$.

We study representations of this algebra in some nondegenerate completed space J. The completeness can be considered as a weak one, or with respect to a Banach norm.

We consider a class of irreducible representations which we call regular. They are Fock representations for the case of Heisenberg algebra and Hilbert space. In general case we have two equivalent definitions of such representations.

Definition 1. *Representation of generalized Heisenberg algebra is regular if operator N has a simple discrete spectrum.*

We start with an eigenvector ψ_λ:

$$N\, \psi_\lambda = \lambda\, \psi_\lambda. \tag{2}$$

The whole set of eigenvectors of N in this case is the set of vectors

$$\begin{array}{ll} \psi_{\lambda+n} = (a^+)^n\, \psi_\lambda; & \psi_{\lambda-n} = a^n\, \psi_\lambda; \\ N\, \psi_{\lambda+n} = (\lambda+n)\, \psi_{\lambda+n}, ; & N\, \psi_{\lambda-n} = (\lambda-n)\, \psi_{\lambda-n}; \end{array} \quad n \in \mathbb{N} \tag{3}$$

It follows from $(1')$ that

$$[a\, a^+,\, N] = [a^+\, a,\, N] = 0 \tag{4}$$

Simplicity of spectrum of N implies that

$$a^+\, a\, \psi_{\lambda+k} = b_k\, \psi_{\lambda+k}; \quad a\, a^+\, \psi_{\lambda+k} = \tilde{b_k}\, \psi_{\lambda+k}. \tag{5}$$

Since $a\, \psi_{\lambda+n} \sim \psi_{\lambda+n-1}$, $a^+\, \psi_{\lambda-n} \sim \psi_{\lambda-n+1}$

$$P_n\,(a, a^+)\, \psi_\lambda = \sum C_k\, \psi_{\lambda+k}, \qquad k \in \mathbb{Z}, \tag{6}$$

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where $P_n(a, a^+)$ is an arbitrary polynomial in a and a^+. Eqs. (5) imply that

$$a^+ a = \varphi(N), \qquad a\, a^+ = \varphi_1(N) \tag{7}$$

From (1′), $a\, a^+ a = a\, \varphi(N) = \varphi_1(N)\, a = a\, \varphi_1(N-1)$, and

$$\varphi_1(N) = \varphi(N+1) \tag{8}$$

In regular case

$$[a, a^+] = \varphi(N+1) - \varphi(N) \tag{9}$$

It should be noted that for Heisenberg algebra $\varphi(N) = N$.

Some steps in the investigation of generalized Heisenberg algebra can be done independently of concrete form of the function $\varphi(N)$.

First, let us construct the space J, starting from the space of finite linear combinations of $\psi_{\lambda+k}$, which we denote as J_0.

J is a closed space, so $J = \bar{J}_0$, but a closure of J_0 is not unique as it is determined by the topology of J_0. It is natural to consider the representations in question in all these spaces as equivalent, as they all contain the dense (weakly or strongly) domain J_0 (see discussion of similar problems in [2]). So we can confine ourselves to the natural closure, using the intrinsic inner product.

Let us point out that N has no neutral eigenvectors, because J and also J_0 are nondegenerate spaces. So we can construct vectors

$$e_k = \frac{\psi_{\lambda+k}}{|<\psi_{\lambda+k}, \psi_{\lambda+k}>|^{\frac{1}{2}}},$$

which satisfy the conditions:

$$<e_k, e_m> = \delta_{k\,m}\, \delta_k, \qquad \delta_k = \pm 1.$$

Dividing the set of e_k on that of positive e_k^+ and negative e_k^- vectors, we can represent J_0 in the form

$$J_0 = J_0^+ \oplus J_0^-, \qquad J_0^+ \perp J_0^-. \tag{10}$$

Thus

$$X = X^+ + X^-; \quad X^\pm = \sum_k C_k^\pm\, e_k^\pm, \quad X^\pm \in J_0^\pm. \tag{10′}$$

Now let us consider the space of all infinite linear combinations e_k, such that $(<X, X>) < \infty$, $X = \sum_0^\infty C_k\, e_k$. This space includes all vectors $X^\pm = \sum_k C_k^\pm\, e_k^\pm$ which satisfy the condition:

$$\sum_k^\infty |\, C_k^\pm\,|^2 < \infty. \tag{11}$$

The condition (11) is sufficient for $(<X, X>) < \infty$.

But this condition also is the necessary one, as the space under consideration contains vectors $X^+ - X^-$ as well as $X^+ + X^-$, and so the vectors $X^\pm$. The natural closure is defined as follows:

$$\bar{J}_0 = \bar{J}_0^+ \oplus \bar{J}_0^-. \tag{12}$$

The algebra in question is defined in some dense domain D, the concrete form of this domain depends on the properties of the function $\varphi(N)$.

It can be shown that any irreducible regular representation is fully determined by spectrum of N, if the function $\varphi(N)$ is fixed. This means that if R and R' are representations of the algebra in question and $Sp\,N = Sp\,N'$, $\varphi(N) = \varphi'(N')$, then there exists a unitary operator V such that $a' = V\, a\, V^+$, $(a')^+ = V\, a^+\, V^+$.

It is sufficient to show, that all scalar products $<\psi_{\lambda+k}, \psi_{\lambda+k}>$ are fully determined by $\varphi(N)$, if λ is fixed.

Really, in accordance with Eqs. (3), (7) and (8)

$$<\psi_{\lambda+n}, \psi_{\lambda+n}> = <\psi_{\lambda+n-1}, a\, a^+\, \psi_{\lambda+n-1}> = \varphi(\lambda+n)\; <\psi_{\lambda+n-1}, \psi_{\lambda+n-1}>, \tag{13}$$

$$< \psi_{\lambda-n}, \psi_{\lambda-n} >=< \psi_{\lambda-n+1}, a^{+} a \, \psi_{\lambda-n+1} >= \varphi(\lambda-n+1) \, < \psi_{\lambda-n+1}, \psi_{\lambda-n+1} > . \tag{14}$$

It follows immediately from eqs. (13) and (14), that

$$< \psi_{\lambda+n}, \psi_{\lambda+n} >= \varphi(\lambda+n) \, \varphi(\lambda+n-1) \cdots \varphi(\lambda+1) \, < \psi_\lambda, \psi_\lambda >, \tag{15}$$

$$< \psi_{\lambda-n}, \psi_{\lambda-n} >= \varphi(\lambda-n+1) \, \varphi(\lambda-n+2) \cdots \varphi(\lambda) \, < \psi_\lambda, \psi_\lambda >, \tag{16}$$

As $< \psi_\lambda, \psi_\lambda >=< \psi'_\lambda, \psi'_\lambda >$, then $< x', y' >=< x, y >$, if

$$x = \sum_{l_1}^{l_2} C_k \, \psi_{\lambda+k}, \quad x' = \sum_{l_1}^{l_2} C_k \, \psi'_{\lambda+k}, \quad y = \sum_{m_1}^{m_2} \tilde{C}_k \, \psi_{\lambda+k}, \quad y' = \sum_{m_1}^{m_2} \tilde{C}_k \, \psi'_{\lambda+k}, \quad l_1, l_2, m_1, m_2 \in \mathbb{Z}$$

Thus we have proved unitary equivalence of R and R' in J_0 and J'_0 and so in J and J'.

Now let us show that in any regular representation there exists operator $U(s) \equiv \exp(i \, s \, N)$, $s \in \mathbb{R}$ with the following properties:

$$\begin{array}{l} U(s) \, U(t) = U(s+t), \\ U(s) \to \mathbb{I}, \quad s \to 0, \quad U(0) = \mathbb{I}, \\ \lim\limits_{s \to 0} \dfrac{U(s) - \mathbb{I}}{s} \, \psi = i \, N \, \psi \quad \text{if} \quad \psi \in D(N) \end{array} \tag{17}$$

In order to see this, it is sufficient to define $U(s)$ on basis vectors as follows:

$$U(s) \, \psi_{\lambda+k} = \exp\left[i \, s \, (\lambda+k)\right] \psi_{\lambda+k} \tag{18}$$

The first condition in (17) is checked directly. Moreover, it is easy to see that $U(s)$ is a bounded operator.

Actually,

$$U(s) \sum C_k \, \psi_{\lambda+k} = \sum C'_k \, \psi_{\lambda+k}, \quad C'_k = \exp\left[i \, s \, (\lambda+k)\right] C_k .$$

To show that N is a generator of the group $U(s)$, we first consider the space J_0. For this space assertion follows immediately from Eq. (18). Since $U(s)$ is bounded, the assertion remains true, if $N \psi$ is a convergent series, or if $\psi \in D(N)$. Then Eq. (18) implies that

$$U(2 \pi k) = \exp\left[i \, 2 \pi k \lambda)\right], \quad k \in \mathbb{Z} \tag{19}$$

It is easy to check that

$$U(s) \, a = \exp\left[-i \, s\right] a \, U(s), \quad U(s) \, a^{+} = \exp\left[i \, s\right] a^{+} \, U(s) \tag{20}$$

So

$$\left[U(2 \pi k), a\right] = \left[U(2 \pi k), a^{+}\right] = 0, \quad k \in \mathbb{Z} \tag{21}$$

Let us point out that we can obtain an eigenvector of N acting by the operator $W_k(2\pi)$ on any vector ψ, where

$$W_k(s) = \int_0^s e^{-i k s'} \, V(s') \, d s', \quad V(s) = \exp(-i \lambda s) \, U(s) \tag{22}$$

Actually,

$$W_k(2\pi) \psi = 2 \pi \, C_k \, \psi_{\lambda+k}, \qquad \psi = \sum_l C_l \, \psi_{\lambda+l}, \tag{23}$$

We have shown the existence of the operator $U(s)$ in any regular representation.

Now let us prove the converse: the existence of operator $U(s)$ implies that the operator N has an eigenvector. Thus, we can define regular representation of a generalized Heisenberg algebra as follows:
Definition 2. *Representation of a generalized Heisenberg algebra is regular, if there exists operator $U(s)$, satisfying conditions (17).*

Remark 1

It is sufficient to suppose that conditions (17) are valid on some domain D, stable with respect to the action of operator $U\,(s),\ s \in \mathbb{R}$.

First, let us point out that $U\,(s)$ is unitary, because N is self-adjoint.

Indeed, from conditions (17) it follows that

$$\frac{d}{ds}\, U\,(s) = i\, N\, U\,(s). \tag{24}$$

So

$$\frac{d}{ds} < U\,(s)\,\psi, U\,(s)\,\psi >= 0 \qquad \forall\, \psi \tag{25}$$

Further, conditions (20) are valid as before, because they are the consequences of the following equalities:

$$\frac{d}{ds}\,(\,U\,(s)\,a\,U\,(-\,s)\,) = i\,U\,(s)\,[N,a]\,U\,(-\,s) = -\,i\,U\,(s)\,a\,U\,(-\,s),$$

$$\frac{d}{ds}\,\left(\,U\,(s)\,a^{+}\,U\,(-\,s)\,\right) = i\,U\,(s)\,[N,a^{+}]\,U\,(-\,s) = i\,U\,(s)\,a^{+}\,U\,(-\,s).$$

Thus, $U\,(2\,\pi\,k)\ \forall\ k\ \in\mathbb{Z}$ commutes with a and a^{+}. Thus $U\,(2\,\pi\,k) = C_k\,\mathbb{I}$ for irreducible representations , where $\mathbb{I}$ is an identical operator. Mathematical details one can find in [2].

Since $U\,(s)$ is unitary

$$U\,(2\,\pi) = \exp\,[i\,2\,\pi\,\lambda)], \qquad \lambda \in \mathbb{R}. \tag{26}$$

Eq. (23) coincides with Eq. (19).

An eigenvector of N can be constructed as before with the help of operator $W_k\,(2\,\pi)$ (see Eq. (22)). Actually, for any $\psi \in D$ $\psi_k = W_k\,(2\,\pi)\,\psi$ is either an eigenvector of $V\,(s)$ for an arbitrary s, or $\psi_k = 0$:

$$V\,(s)\,\psi_k = \int\limits_0^{2\,\pi} e^{-\,i\,k\,s'}\,V\,(s+s')\,\psi\,d\,s' = e^{i\,k\,s}\int\limits_s^{2\pi+s} e^{-i\,k\,s'}\,V\,(s')\,\psi\,d\,s' == e^{i\,k\,s}\,\psi_k$$

$(V\,(2\,\pi)) = \mathbb{I}$ in accordance with (26).

The last step in the proof of Eq. (4) is the passage to the limit $s \to 0$.

Remark 2

The existence of operator $W_k\,(s)$ is a consequence of the completeness properties of the space J. It is sufficient for $J$to be weakly completed [2]. If J is a Banach space, $W_k\,(s)$ exists because

$$|\,V\,(s'+\Delta) - V\,(s')\,| < \epsilon\,(\Delta)$$

uniformly for any $s' \in (0,s)$. Actually, in the opposite case, there exists a sequence $s_n,\ s_n \to \tilde{s},\ n \to \infty,\ \tilde{s} \in (0,s)$ such that $V\,(s_n) \not\to V\,(\tilde{s})$ in contradiction with the continuity of $V\,(s)$.

Remark 3

For any $\psi \in D$ there exists k_0, such that $\psi_{k_0} \neq 0$. At this point we use the nondegeneracy of the space J. If $\psi_k = 0\ \forall\ k$, then for any vector Φ

$$\int\limits_0^{2\,\pi} e^{-\,i\,k\,s}\,T\,(s)\,d\,s = 0, \quad \text{where} \quad T\,(s) =< \Phi, V\,(s)\,\Psi > .$$

$T\,(s)$ is a periodic function $(T\,(2\,\pi) = T\,(0))$ with all coefficients of its Fourier series equal to zero. Thus $T\,(s) = 0$ and $\Psi \perp \Phi\ \forall\ \Phi$, which implies $\Psi = 0$.

Let us point out that eigenvectors of N can belong not to D, but only to its closure $\bar{D}$. This implies that D is the domain of rapidly convergent infinite sequences $\sum C_k\,\psi_{\lambda+k}$. In the case of Heisenberg algebra , C_k satisfies the condition: $C_k\,k^n \to 0\ \forall\ n,\ k \to \infty$.

For a simple eigenvector ψ_λ we can repeat our scheme of construction of the whole set of eigenvectors of N.

In conclusion let us enumerate all kinds of regular irreducible representations of the algebra in question.

1. Spectrum bounded from below.

In this case there exists n such that $\varphi(\lambda - n) = 0$ and, according to Eq. (16), $< \psi_{\lambda-n-1}, \psi_{\lambda-n-1} >= 0$. As it was already mentioned above, owing to nondegeneracy of J N has no neutral vectors, and so $\psi_{\lambda-n-1} = 0$. If $Sp\, N$ is not bounded from above, then

$$Sp\, N = \lambda - n + m, \qquad m \in \mathbb{N}.$$

Because representation in question is irreducible, each eigenvector of N can be regarded as cyclic. It follows directly from our consideration that, starting with any eigenvector of N, we come to the same set of eigenvectors of N and so to the same space J_0. Really, we can start with any $X \in J_0$. Thus, such a representation is irreducible by construction. It is convenient to choose as cyclic vector $\psi_{\lambda-n}$. We can define such representations as Fock representations , because

$$\psi_{\lambda-n-1} = a\, \psi_{\lambda-n} = 0, \tag{27}$$

2. Spectrum bounded from above.

In this case $\psi_{\lambda+m+1} = 0$ for some m, or equivalent

$$a^+\, \psi_{\lambda+m} = 0, \tag{28}$$

owing to Eq. (15). It is natural to name this case as anti-Fock case.

We can consider $\psi_{\lambda+m}$ as a cyclic vector. Evidently, $Sp\, N = \lambda + m - n,\ n \in \mathbb{N}$.

3. Finite spectrum.

In this case Eqs. (27) and (28) are valid simultaneously.
$Sp\, N = \lambda - n,\ \lambda - n + 1,\ \cdots\ \lambda + m$. Such kind of $Sp\, N$ is not realized in Heisenberg algebra, but it can exist in its q- and R-deformations.

4. Unbounded spectrum.

In this case $\varphi(\lambda + k) \neq 0\ \forall\ k$, so $a\, \psi_{\lambda+k} \neq 0,\ a^+\, \psi_{\lambda+k} \neq 0,\ \forall\ k$.

Thus we see that investigation of concrete cases of generalized Heisenberg algebra is reduced to the study of the properties of the functions $\varphi(\lambda + k)$.

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