Dynamical Quantization

Vasily E. Tarasov

Institute of Nuclear Physics,
Moscow State University, Moscow 119899, Russia
E-mail: tarasov@theory.sinp.msu.ru

Abstract

A generalization of Weyl quantization which maps a dynamical operator to a dynamical superoperator is suggested. Weyl quantization of dynamical operator, which cannot be represented as Poisson bracket with some function, is considered. The usual canonical quantization is a specific case of suggested quantization. This approach allows to define consistent quantization procedure for non-Hamiltonian and dissipative systems. Examples of the harmonic oscillator with friction, the Fokker-Planck-type system and Lorenz-type are considered.

1 Introduction

The quantization of dissipative and non-Hamiltonian classical systems is of strong theoretical interest. As a rule, any microscopic system is always embedded in some (macroscopic) environment and therefore it is never really isolated. Frequently, the relevant environment is in principle unobservable or it is unknown [1]-[4]. This would render theory of dissipative and non-Hamiltonian systems a fundamental generalization of quantum mechanics [5].

We can divide the most frequent methods of quantization of dissipative and non-Hamiltonian systems into two groups. The first method uses a procedure of doubling of phase-space dimension [6]-[8]. The second method consists in using an explicitly time-dependent Hamiltonian [9]-[16].

H. Bateman has shown [6] that in order to use the usual canonical quantization methods a procedure of doubling of phase-space dimension is required. To apply the usual canonical quantization scheme to dissipative and non-Hamiltonian systems, one can double the numbers of degrees of freedom, so as to deal with an effective isolated system. The new degrees of freedom may be assumed to represent by collective degrees of freedom of the bath, which absorb the energy dissipated by the dissipative system [7, 8].

P. Cardirola [9] and E. Kanai [10] have shown that it may be possible to put the equation of motion for dissipative system into time-dependent Hamiltonian form and then quantize them in the usual way [9]-[16]. However, the corresponding canonical commutation relations violate the uncertainty principle [14]. The reason for this violation would appear from the explicit dependence of Hamiltonian and momentum on the time.

To construct the quantization of dissipative (non-Hamiltonian) systems consistently, it is possible to exceed the limits of Lie algebras and groups. The condition of self-consistency for the quantization of dissipative systems requires the application of Valya algebra [17, 18]. Unfortunately, this algebra and its representation have not been thoroughly studied.

Note [19, 16] that R. Feynman wanted to develop a procedure to quantize classical equation of motion without resort to a Hamiltonian. It is interesting to quantize a classical system without direct reference to a Hamiltonian. A general classical system is most easily defined in terms of its equations of motion. In general case it is difficult to determine whether a Hamiltonian exists, whether it is unique if it does exist [20, 21]. Therefore, quantization that bypasses direct reference to a Poisson bracket with some Hamiltonian may have practical advantages.

Canonical quantization defines a map of real functions into self-adjoint operators [22, 23]. A classical observable is described by some real function $A(q,p)$ from a function space $\mathcal{M}$. Quantization of this function leads to self-adjoint operator $\hat{A}(\hat{q},\hat{p})$ from some operator space $\hat{\mathcal{M}}$. Note that canonical quantization of the Poisson bracket $\{\ .\ ,\ .\ \}$ leads to commutator $(-i/\hbar)[\ .\ ,\ .\ ]$. Classical state can be described by non-negative-normed function $\rho(q,p)$ called density distribution function. Quantization of a function $\rho(q,p)$ leads to non-negative self-adjoint operator $\hat{\rho}$ of trace class called matrix density operator. This description allows to consider a state as a special observable.

Time evolution of an observable $A_t(q,p)$ and a state $\rho_t(q,p)$ in classical mechanics are described by differential equations on a function space $\mathcal{M}$:

$$\frac{d}{dt}A_t(q,p)=\mathcal{L}A_t(q,p) \ , \quad \frac{d}{dt}\rho_t(q,p)=\Lambda\rho_t(q,p) \ .$$

The operators $\mathcal{L}$ and Λ, which act on the elements of function space $\mathcal{M}$, define dynamics. These operators are infinitesimal generators of dynamical semigroups and are called dynamical operators. The first equation describes evolution of an observable in the Hamilton picture, and the second equation describes evolution of a state in the Liouville picture.

Dynamics of an observable $\hat{A}_t(\hat{q},\hat{p})$ and a state $\hat{\rho}_t$ in quantum mechanics are described by differential equations on an operator space $\hat{\mathcal{M}}$:

$$\frac{d}{dt}\hat{A}_t(\hat{q},\hat{p})=\hat{\mathcal{L}}\hat{A}_t(\hat{q},\hat{p}) \ , \quad \frac{d}{dt}\hat{\rho}_t=\hat{\Lambda}\hat{\rho}_t \ .$$

Here $\hat{\mathcal{L}}$ and $\hat{\Lambda}$ are superoperators, i.e. operators act on operators. These superoperators are infinitesimal generators of quantum dynamical semigroups [26, 27, 28]. The first equation describes dynamics in the Heisenberg picture, and the second - in the Schroedinger picture. It is easy to see that quantization of the dynamical operators $\mathcal{L}$ and Λ leads to dynamical superoperators $\hat{\mathcal{L}}$ and $\hat{\Lambda}$. Therefore, generalization of canonical quantization must map operators into superoperators and map classical dynamical equations into quantum dynamical equations. This quantization procedure called dynamical quantization.

The usual method of quantization is applied to classical systems, where the dynamical operators have the forms $\mathcal{L}A(q,p)=\{A(q,p),H(q,p)\}$ and $\Lambda\rho(q,p)=-\{\rho(q,p),H(q,p)\}$. Here the function $H(q,p)$ is an observable which characterizes dynamics and is called the Hamilton function. Quantization of a dynamical operator which can be represented as Poisson bracket with a function is defined by the usual canonical quantization. Canonical quantization of real functions $A(q,p)$ and $H(q,p)$ leads to self-adjoint operators $\hat{A}(\hat{q},\hat{p})$ and $\hat{H}(\hat{q},\hat{p})$. Quantization of the Poisson bracket $\{A(q,p),H(q,p)\}$ leads to commutator $(i/\hbar)[\hat{H}(\hat{q},\hat{p}),\hat{A}(\hat{q},\hat{p})]$. Therefore quantization of these dynamical operators is uniquely defined by the usual canonical quantization.

Quantization of a dissipative and non-Hamiltonian classical system by using Hamiltonian meets ambiguities which follow from the problems of variational description of these systems [20]-[21]. Quantization of dissipative and non-Hamiltonian systems is not defined by the usual canonical quantization. Therefore, it is necessary to consider some generalization of canonical quantization. This generalized procedure must define a map of operator into superoperator [29]. The usual canonical quantization must be a specific case of generalized quantization, for quantization of operator of multiplication on a function.

In this paper Weyl quantization of dissipative and non-Hamiltonian classical systems is considered. Generalization of Weyl quantization, which maps an evolution equation on a function space into an evolution equation on an operator space, is suggested. An analysis of generalized quantization is performed for operator, which cannot be represented as the Poisson bracket with some Hamilton function.

2 Canonical Weyl Quantization

Let us consider main points of the usual method of Weyl quantization [22, 24, 30, 31]. Let q^k are canonical coordinates and p^k are canonical momentums, where $k=1,...,n$. The basis of the space $\mathcal{M}$ of functions $A(q,p)$ is defined by functions

$$W(a,b,q,p)=e^{\frac{i}{\hbar}(aq+bp)} \ , \quad aq=\sum_{k=1}^{n}a_kq^k \ . \tag{1}$$

Quantization transforms coordinates q^k and momentums p^k to operators $\hat{q}^k$ and $\hat{p}^k$. Weyl quantization of the basis functions (1) leads to the Weyl operators

$$\hat{W}(a,b,\hat{q},\hat{p})=e^{\frac{i}{\hbar}(a\hat{q}+b\hat{p})} \ , \quad a\hat{q}=\sum_{k=1}^{n}a_k\hat{q}^k \ . \tag{2}$$

The Weyl operators form a basis of the operator space $\hat{\mathcal{M}}$. Classical observable, characterized by the function $A(q,p)$, can be represented in the form

$$A(q,p) = \frac{1}{(2\pi\hbar)^n}\int \tilde{A}(a,b)W(a,b,q,p)d^n a d^n b \ , \tag{3}$$

where $\tilde{A}(a,b)$ is the Fourier image of the function $A(q,p)$. Quantum observable $\hat{A}(\hat{q},\hat{p})$ which corresponds to $A(q,p)$ is defined by formula

$$\hat{A}(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^n}\int \tilde{A}(a,b)\hat{W}(a,b,\hat{q},\hat{p})d^n a d^n b \ . \tag{4}$$

This formula can be considered as an operator expansion for $\hat{A}(\hat{q},\hat{p})$ in the operator basis (2). The direct and inverse Fourier transformations allow to write the formula (4) for the operator $\hat{A}(\hat{q},\hat{p})$ as

$$\hat{A}(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^{2n}}\int A(q,p)\hat{W}(a,b,\hat{q}-q\hat{I},\hat{p}-p\hat{I})d^n a d^n b d^n q d^n p \ . \tag{5}$$

The function $A(q,p)$ is called the Weyl symbol of the operator $\hat{A}(\hat{q},\hat{p})$. Canonical quantization defined by (5) is called the Weyl quantization.

Algebraic structures can be defined on the set of obrervables. Lie algebra, Jordan algebra and C^*-algebra are usually considered on the spaces $\mathcal{M}$ and $\hat{\mathcal{M}}$.

Lie algebra $L(\mathcal{M})$ on the set $\mathcal{M}$ is defined by Poisson bracket

$$\{A(q,p),B(q,p)\} = \sum_{k=1}^{n}\Big(\frac{\partial A(q,p)}{\partial q^k}\frac{\partial B(q,p)}{\partial p^k} - \frac{\partial A(q,p)}{\partial p^k}\frac{\partial B(q,p)}{\partial q^k}\Big) \ . \tag{6}$$

Quantization of the Poisson bracket leads to self-adjoint commutator

$$\frac{1}{i\hbar}[\hat{A}(\hat{q},\hat{p}),\hat{B}(\hat{q},\hat{p})] = \frac{1}{i\hbar}\Big(\hat{A}(\hat{q},\hat{p})\hat{B}(\hat{q},\hat{p}) - \hat{B}(\hat{q},\hat{p})\hat{A}(\hat{q},\hat{p})\Big) \ . \tag{7}$$

The commutator defines Lie algebra $\hat{L}(\hat{\mathcal{M}})$ on the set $\hat{\mathcal{M}}$. Leibnitz rule is satisfied for the Poisson brackets. As a result, the Poisson brackets are defined by basis Poisson brackets for canonical coordinates and momentums

$$\{q^k,q^m\}=0, \quad \{p^k,p^m\}=0, \quad \{q^k,p^m\}=\delta_{km} \ .$$

Quantization of these relations lead to the canonical commutation relations

$$[\hat{q}^k,\hat{q}^m]=0, \quad [\hat{p}^k,\hat{p}^m]=0, \quad [\hat{q}^k,\hat{p}^m]=i\hbar\delta_{km}\hat{I} \ . \tag{8}$$

These relations define $(2n+1)$-parametric Lie algebra $\hat{L}(\hat{\mathcal{M}})$, called Heisenberg algebra.

Jordan algebra $J(\mathcal{M})$ for the set $\mathcal{M}$ is defined by the multiplication $A \circ B$ which coincides with the usual associative multiplication of functions. Weyl quantization of the Jordan algebra $J(\mathcal{M})$ leads to the operator special Jordan algebra $\hat{J}(\hat{\mathcal{M}})$ with multiplication

$$[\hat{A},\hat{B}]_+ = \hat{A}\circ\hat{B} = \frac{1}{4}[(\hat{A}+\hat{B})^2 - (\hat{A}-\hat{B})^2] \ .$$

Jordan algebra for classical observables is associative algebra, that is all associators are equal to zero

$$(A\circ B)\circ C - A\circ(B\circ C) = 0 \ .$$

In general case Jordan algebra associator for quantum observables is not equal to zero

$$(\hat{A}\circ\hat{B})\circ\hat{C} - \hat{A}\circ(\hat{B}\circ\hat{C}) = \frac{1}{4}[\hat{B},[\hat{C},\hat{A}]] \ . \tag{9}$$

This nonassociativity of the operator Jordan algebra $\hat{J}(\hat{\mathcal{M}})$ leads to the ambiguity of canonical quantization. The arbitrariness is connected with ordering of noncommutative opetators. The Weyl operator (2) in formula (6) leads to Weyl quantization. Another basis operator [24, 22] leads to different quantization scheme.

C^*-algebra can be defined on the set of quantum observables described by the bounded linear operators. The operator which is a result of associative multiplication of the self-adjoint operators is not self-adjoint operator. Therefore quantization of multiplication of classical observables does not leads to multiplication of the correspondent quantum observables. Universal enveloping algebra $\hat{U}(\hat{L})$ for the Lie algebra $\hat{L}(\hat{\mathcal{M}})$ which is generated by commutation relations (8) usually is considered as associative algebra [30, 31].

Let us consider a classical dynamical system defined by Hamilton function $H(q,p)$. Usually the quantization procedure is applied to classical systems with dynamical operator

$$\mathcal{L} = -\{H(q,p),\ .\ \} = -\sum_{k=1}^{n}\Big(\frac{\partial H(q,p)}{\partial q^k}\frac{\partial}{\partial p^k} - \frac{\partial H(q,p)}{\partial p^k}\frac{\partial}{\partial q^k}\Big)\ . \tag{10}$$

Here $H(q,p)$ is an observable which defines dynamics of a classical system. The observable $H(q,p)$ is called the Hamilton function. The time evolution of a classical observable is described by

$$\frac{d}{dt}A_t(q,p) = \{A_t(q,p), H(q,p)\}\ . \tag{11}$$

If the dynamical operator has the form (10), then system is Hamiltonian system. Weyl quantization of the functions $A_t(q,p)$ and $H(q,p)$ leads to operators $\hat{A}_t(\hat{q},\hat{p})$ and $\hat{H}(\hat{q},\hat{p})$. Weyl quantization of Poisson bracket $\{A_t(q,p), H(q,p)\}$ leads to the commutator $(i/\hbar)[\hat{H}(\hat{q},\hat{p}), \hat{A}_t(\hat{q},\hat{p})]$. Quantization of the equation (11) leads to the Heisenberg equation

$$\frac{d}{dt}\hat{A}_t(\hat{q},\hat{p}) = \frac{i}{\hbar}[\hat{H}(\hat{q},\hat{p}), \hat{A}_t(\hat{q},\hat{p})]\ .$$

Therefore quantization of the dynamical operator (10) leads to superoperator [30]

$$\hat{\mathcal{L}} = \frac{i}{\hbar}[\hat{H}(\hat{q},\hat{p}),\ .\ ] = \frac{i}{\hbar}(\hat{H}^l(\hat{q},\hat{p}) - \hat{H}^r(\hat{q},\hat{p})) \tag{12}$$

Here left and right superoperators $\hat{H}^l(\hat{q},\hat{p})$ and $\hat{H}^r(\hat{q},\hat{p})$ correspond to Hamilton operator $\hat{H}(\hat{q},\hat{p})$. These superoperators are defined by formulas

$$\hat{H}^l\hat{A} = \hat{H}\hat{A}\ , \quad \hat{H}^r\hat{A} = \hat{A}\hat{H}\ .$$

Quantization of dynamical operator, which can be represented as Poisson bracket with a function, is defined by canonical quantization. Therefore usual Weyl quantization of Hamiltonian systems is completely defined by the usual method of quantization.

3 General Dynamical System

Let us consider the time evolution of classical observable $A_t(q,p)$, described by the general differential equation

$$\frac{d}{dt}A_t(q,p) = \mathcal{L}(q,p,\partial_q,\partial_p)A_t(q,p)\ .$$

Here

$$\partial_q = \frac{\partial}{\partial q}\ , \quad \partial_p = \frac{\partial}{\partial p}\ ,$$

and $\mathcal{L}(q,p,\partial_q,\partial_p)$ is an operator on the function space $\mathcal{M}$. In general case the operator $\mathcal{L}$ cannot be expressed by Poisson bracket with a function $H(q,p)$. We would like to generalize the quantization procedure from the dynamical operators (10) to general operators $\mathcal{L}(q,p,\partial_q,\partial_p)$. In order to describe generalized quantization we must define the operator $\mathcal{L}(q,p,\partial_q,\partial_p)$ using some operator basis. For simplicity, we assume that operator $\mathcal{L}(q,p,\partial_q,\partial_p)$ is a bounded operator.

Let us define the basis operators which generate the dynamical operator $\mathcal{L} = \mathcal{L}(q, p, \partial_q, \partial_p)$. Operators Q_1^k and Q_2^k are operators of multiplication on q^k and p^k. Operator P_1^k and P_2^k are self-adjoint differential operator with respect to q^k and p^k, that is $P_1^k = -i\partial/\partial q^k$ and $P_2^k = -i\partial/\partial p^k$. These operators obey the conditions:

1. $Q_1^k 1 = q^k$, $Q_2^k 1 = p^k$ and $P_1^k 1 = 0$, $P_2^k 1 = 0$.
2. $(Q_{1,2}^k)^* = Q_{1,2}^k$, $(P_{1,2}^k)^* = P_{1,2}^k$.
3. $P_{1,2}^k(A \circ B) = (P_{1,2}^k A) \circ B + A \circ (P_{1,2}^k B)$.
4. $[Q_{1,2}^k, P_{1,2}^m] = i\delta_{km}$, $[Q_{1,2}^k, P_{2,1}^m] = 0$.
5. $[Q_{1,2}^k, Q_{1,2}^m] = 0$, $[P_{1,2}^k, P_{1,2}^m] = 0$.

Conjugation operation $*$ is defined with respect to the usual scalar product of function space. Commutation relations for the operators $P_{1,2}^k$ and $Q_{1,2}^k$ define $(4n+1)$-parametric Lie algebra. These relations are analogous to canonical commutation relations (8) for $\hat{q}^k$ and $\hat{p}^k$ with double numbers of degrees of freedom.

Operators $Q_{1,2}^k$ and $P_{1,2}^k$ allow to introduce operator basis

$$V(a_1, a_2, b_1, b_2, Q_1, Q_2, P_1, P_2) = exp\{i(a_1 Q_1 + a_2 Q_2 + b_1 P_1 + b_2 P_2)\} , \tag{13}$$

for the space $\mathcal{A}(\mathcal{M})$ of dynamical operators. These basis operators are analogous to the Weyl operators (2). Note that basis functions (1) can be derived from the operators (13) by the formula

$$W(a, b, q, p) = V((a/\hbar), (b/\hbar), 0, 0, Q_1, Q_2, P_1, P_2) 1 .$$

The algebra $\mathcal{A}(\mathcal{M})$ of bounded dynamical operators can be defined as C^*-algebra, generated by $Q_{1,2}^k$ and $P_{1,2}^k$. It contains all operators (13) and is closed for linear combinations of (13) in operator norm topology. A dynamical operator $\mathcal{L}$ can be defined as an operator function of basis operators $Q_{1,2}^k$ and $P_{1,2}^k$:

$$\mathcal{L}(Q_1, Q_2, P_1, P_2) = \frac{1}{(2\pi)^{2n}} \int \tilde{L}(a_1, a_2, b_1, b_2) e^{i(a_1 Q_1 + a_2 Q_2 + b_1 P_1 + b_2 P_2)} d^n a_1 d^n a_2 d^n b_1 d^n b_2 , \tag{14}$$

where $\tilde{L}(a_1, a_2, b_1, b_2)$ is square-integrable function of real variables $a_{1,2}$ and $b_{1,2}$. The function $\tilde{L}(a_1, a_2, b_1, b_2)$ is Fourier image of the symbol of operator $\mathcal{L} = \mathcal{L}(q, p, \partial_q, \partial_p)$. The set of bounded operators $\mathcal{L}(Q_1, Q_2, P_1, P_2)$ and their uniformly limits forms the algebra $\mathcal{A}(\mathcal{M})$ of dynamical operators.

4 Weyl Quantization of Basis Operators

To define the quantization of the operator $\hat{\mathcal{L}}$ which corresponds to operator $\mathcal{L}$ we need to describe quantization of the operators Q^k and P^k. Let us require that the superoperators $\hat{Q}^k$ and $\hat{P}^k$ satisfy the relations which are the quantum analogs to the relations for the operators Q^k and P^k:

1. $\hat{Q}_1^k \hat{I} = \hat{q}^k$, $\hat{Q}_2^k \hat{I} = \hat{p}^k$, and $\hat{P}_{1,2}^k \hat{I} = 0$.
2. $(\hat{Q}_{1,2}^k)^* = \hat{Q}_{1,2}^k$, $(\hat{P}_{1,2}^k)^* = \hat{P}_{1,2}^k$.
3. $\hat{P}_{1,2}^k(\hat{A} \circ \hat{B}) = (\hat{P}_{1,2}^k \hat{A}) \circ \hat{B} + \hat{A} \circ (\hat{P}_{1,2}^k \hat{B})$.
4. $[\hat{Q}_{1,2}^k, \hat{P}_{1,2}^m] = i\delta_{km}\hat{I}$, $[\hat{Q}_{1,2}^k, \hat{P}_{2,1}^m] = 0$.
5. $[\hat{Q}_{1,2}^k, \hat{Q}_{1,2}^m] = 0$, $[\hat{P}_{1,2}^k, \hat{P}_{1,2}^m] = 0$.

Superoperator $\hat{\mathcal{L}}$ is called self-adjoint, if the relation $< \hat{\mathcal{L}}\hat{A}|\hat{B} > = < \hat{A}|\hat{\mathcal{L}}\hat{B} >$ is satisfied. The scalar product $< \hat{A}|\hat{B} >$ on the operator space $\mathcal{M}$ is defined by $< \hat{A}|\hat{B} > \equiv Sp[\hat{A}^* \hat{B}]$. An operator space with this scalar product is called Liouville space [30, 31].

To define a quantization of the operator $P_{1,2}^k$ we use the relations

$$P_1^k A(q,p) = -i\frac{\partial}{\partial q^k} A(q,p) = i\{p^k, A(q,p)\} , \quad P_2^k A(q,p) = -i\frac{\partial}{\partial p^k} A(q,p) = -i\{q^k, A(q,p)\} ,$$

Weyl quantization [25] leads to the expressions

$$\hat{P}_1^k \hat{A}(\hat{q},\hat{p}) = \frac{1}{\hbar}[\hat{p}^k, \hat{A}(\hat{q},\hat{p})] , \quad \hat{P}_2^k \hat{A}(\hat{q},\hat{p}) = -\frac{1}{\hbar}[\hat{q}^k, \hat{A}(\hat{q},\hat{p})] .$$

As a result, we obtain

$$\hat{P}_1^k = \frac{1}{\hbar}[\hat{p}^k,\ .\ ] = \frac{1}{\hbar}((\hat{p}^k)^l - (\hat{p}^k)^r)\ , \quad \hat{P}_2^k = -\frac{1}{\hbar}[\hat{q}^k,\ .\ ] = -\frac{1}{\hbar}((\hat{q}^k)^l - (\hat{q}^k)^r)\ . \tag{15}$$

Here we use superoperators $(\hat{q}^k)^l$, $(\hat{q}^k)^r$ and $(\hat{p}^k)^l$, $(\hat{p}^k)^r$ which satisfy the non-zero commutation relations

$$[(\hat{q}^k)^l, (\hat{p}^m)^l] = i\hbar\delta_{km}\hat{I}, \quad [(\hat{q}^k)^r, (\hat{p}^m)^r] = -i\hbar\delta_{km}\hat{I}.$$

The other commutation relations are equal to zero. These relations follow from canonical commutation relations (8).

Let us define a quantization of the operator $Q_{1,2}^k$. It is known [25, 22, 23] that Weyl quantization of the expressions $q^k \circ A(q,p)$ and $p^k \circ A(q,p)$ leads to $\hat{q}^k \circ \hat{A}(\hat{q},\hat{p})$ and $\hat{p}^k \circ \hat{A}(\hat{q},\hat{p})$. Therefore superoperators $\hat{Q}_{1,2}^k$ are defined by

$$\hat{Q}_1^k = [\hat{q}^k,\ .\ ]_+ = \frac{1}{2}((\hat{q}^k)^l + (\hat{q}^k)^r)\ , \quad \hat{Q}_2^k = [\hat{p}^k,\ .\ ]_+ = \frac{1}{2}((\hat{p}^k)^l + (\hat{p}^k)^r)\ , \tag{16}$$

where $\hat{Q}_1^k\hat{A} = \hat{q}^k \circ \hat{A}$ and $\hat{Q}_2^k\hat{A} = \hat{p}^k \circ \hat{A}$.

Weyl quantization of the basis operators (13) leads to the basis superoperators

$$\hat{V}(a_1, a_2, b_1, b_2, \hat{Q}_1, \hat{Q}_2, \hat{P}_1, \hat{P}_2) = exp\{i(a_1\hat{Q} + a_2\hat{Q}_2 + b_1\hat{P}_1 + b_2\hat{P}_2)\}\ . \tag{17}$$

5 Weyl Quantization of Operator Function

Let us consider the dynamical operator $\mathcal{L}$ as a function of the basis operators $Q_{1,2}^k$ and $P_{1,2}^k$. Generalized Weyl quantization can defined as a map from dynamical operator space $A(\mathcal{M})$ to dynamical superoperator space $\hat{A}(\hat{\mathcal{M}})$. This quantization of the operator

$$\mathcal{L}(Q_1, Q_2, P_1, P_2) = \frac{1}{(2\pi)^{2n}} \int \tilde{L}(a_1, a_2, b_1, b_2) e^{i(a_1Q_1 + a_2Q_2 + b_1P_1 + b_2P_2)} d^n a_1 d^n a_2 d^n b_1 d^n b_2\ ,$$

leads to the corresponding superoperator

$$\hat{\mathcal{L}}(\hat{Q}_1, \hat{Q}_2, \hat{P}_1, \hat{P}_2) = \frac{1}{(2\pi)^{2n}} \int \tilde{L}(a_1, a_2, b_1, b_2) e^{i(a_1\hat{Q}_1 + a_2\hat{Q}_2 + b_1\hat{P}_1 + b_2\hat{P}_2)} d^n a_1 d^n a_2 d^n b_1 d^n b_2\ . \tag{18}$$

If the function $\tilde{L}(a_1, a_2, b_1, b_2)$ is connected with Fourier image $\tilde{A}(a_1, a_2)$ of the function $A(q,p)$ by the relation

$$\tilde{L}(a_1, a_2, b_1, b_2) = (2\pi)^n \delta(b_1)\delta(b_2)\tilde{A}(a_1, a_2)\ ,$$

then formula (18) defines the usual Weyl quantization of the function $A(q,p) = \mathcal{L}(Q_1, Q_2, P_1, P_2)1$ by the relation

$$\hat{A}(\hat{q},\hat{p}) = \hat{\mathcal{L}}(\hat{Q}_1, \hat{Q}_2, \hat{P}_1, \hat{P}_2)\hat{I}\ .$$

Here we use $\hat{Q}_1^k\hat{I} = \hat{q}^k$ and $\hat{Q}_2^k\hat{I} = \hat{p}^k$. Therefore the usual Weyl quantization is a spesific case of suggested quantization procedure.

Superoperators $\hat{Q}_{1,2}^k$ and $\hat{P}_{1,2}^k$ can be represented by $(\hat{q}^k)^l$, $(\hat{q}^k)^r$ and $(\hat{p}^k)^l$, $(\hat{p}^k)^r$. Formula (18) is written in the form

$$\hat{\mathcal{L}}(\hat{q}^l, \hat{q}^r, \hat{p}^l, \hat{p}^r) = \frac{1}{(2\pi)^{2n}} \int L(a_1, a_2, b_1, b_2) W^l(a_1, a_2, \hat{q}, \hat{p}) W^r(b_1, b_2, \hat{q}, \hat{p}) d^n a_1 d^n a_2 d^n b_1 d^n b_2\ .$$

Here $W^l(a, b, \hat{q}, \hat{p})$ and $W^l(a, b, \hat{q}, \hat{p})$ are left and right superoperators corresponding to the Weyl operator (2). These superoperators can be defined by

$$W^l(a, b, \hat{q}, \hat{p}) = W(a, b, \hat{q}^l, \hat{p}^l)\ , \quad W^r(a, b, \hat{q}, \hat{p}) = W(a, b, \hat{q}^r, \hat{p}^r)\ .$$

We can derive [29] a relation which represents the superoperator $\hat{\mathcal{L}}$ by operator $\mathcal{L}$. Let us write the analog of the relation (5) between an operator $\hat{A}$ and a function A. To simplify formulas, we introduce new notations. Let X^s, where $s = 1, ..., 4n$, denote the operators $Q^k_{1,2}$ and $P^k_{1,2}$, where $k = 1, ..., n$, that is

$$X^{2k-1} = q^k \ , \quad X^{2k} = p^k \ , \quad X^{2k-1+2n} = -i\frac{\partial}{\partial q^k} \ , \quad X^{2k+2n} = -i\frac{\partial}{\partial p^k} \ .$$

Let us denote the parameters $a^k_{1,2}$ and $b^k_{1,2}$, where $k = 1, ..., n$, by z^s, where $s = 1, ..., 4n$. Then formula (14) can be rewritten by

$$\mathcal{L} = \frac{1}{(2\pi)^{2n}} \int L(z) e^{izX} d^{4n}z \ .$$

The formula (18) for the superoperator $\hat{\mathcal{L}}$ is written in the form

$$\hat{\mathcal{L}} = \frac{1}{(2\pi)^{2n}} \int L(z) e^{iz\hat{X}} d^{4n}z \ .$$

The result relation which represents the superoperator $\hat{\mathcal{L}}$ by operator $\mathcal{L}$ can be written [29] in the form

$$\hat{\mathcal{L}} = \frac{1}{(2\pi)^{4n}} \int e^{-i\alpha(z+z')} e^{iz\hat{X}} Sp[\mathcal{L}e^{iz'X}] d^{4n}z d^{4n}\alpha d^{4n}z' \ . \qquad (19)$$

6 Harmonic oscillator with friction

Let us consider n-dimensional oscillator with friction $F^k_{fric} = -\alpha_{km}p^m - \beta_{kms}p^m p^s$. The time evolution equation for this oscillator has the form

$$\frac{d}{dt}q^k = \frac{1}{m}p^k \ , \quad \frac{d}{dt}p^k = -(m\omega^2 q^k + \alpha_{km}p^m + \beta_{kms}p^m p^s) \ , \qquad (20)$$

where $k, m, s = 1, ..., n$. If $n = 3$, $\omega = 0$ and nonzero coefficients are

$$\alpha_{11} = 10, \ \alpha_{12} = -10, \ \alpha_{21} = -28, \ \alpha_{22} = 1, \ \alpha_{33} = \frac{8}{3} \ ,$$

$$\beta_{213} = \beta_{231} = 0.5, \ \beta_{312} = \beta_{321} = -0.5 \ ,$$

then we have Lorenz system [32] with respect to $x = p^1$, $y = p^2$ and $z = p^3$. If nonzero coefficients are

$$\alpha_{12} = \alpha_{13} = 1, \ \alpha_{21} = -1, \ \alpha_{22} = -0.2, \ \alpha_{31} = -0.2, \ \alpha_{33} = 5.7, \quad \beta_{313} = \beta_{331} = -0.5 \ ,$$

then we have Rossler-like sytem [33]. If nonzero coefficients are

$$\alpha_{11} = 0.4, \ \alpha_{12} = -1, \ \alpha_{21} = 1, \ \alpha_{22} = 0.4, \ \alpha_{33} = -\alpha = -0.175 \ ,$$

$$\beta_{123} = \beta_{132} = -5, \ \beta_{213} = \beta_{231} = -2.5, \ \beta_{312} = \beta_{321} = 2.5 \ ,$$

then we obtain Newton-Leipnik system [34].

The dynamical equation for the classical observable $A_t(q, p)$ is written

$$\frac{d}{dt}A_t(q, p) = \mathcal{L}(q, p, \partial_q, \partial_p) A_t(q, p) \ .$$

Differentiation of the function $A_t(q, p)$ and equation (20) give

$$\frac{dA_t(q, p)}{dt} = \frac{1}{m}p^k \frac{\partial A_t(q, p)}{\partial q^k} - (m\omega^2 q^k + \alpha_{km}p^m + \beta_{kms}p^m p^s) \frac{\partial A_t(q, p)}{\partial p^k} \ . \qquad (21)$$

Dynamical operator $\mathcal{L}(q, p, \partial_q, \partial_p)$ for system (20) has the form

$$\mathcal{L}(q, p, \partial_q, \partial_p) = \frac{1}{m}p^k \frac{\partial}{\partial q^k} - (m\omega^2 q^k + \alpha_{km}p^m + \beta_{kms}p^m p^s) \frac{\partial}{\partial p^k} \ . \qquad (22)$$

This operator can be rewritten in the form

$$\mathcal{L}(Q_1,Q_2,P_1,P_2) = \frac{i}{m} Q_2^k P_1^k - i(m\omega^2 Q_1^k + \alpha_{km} Q_2^m + \beta_{kms} Q_2^m Q_2^s) P_2^k \ . \tag{23}$$

If we consider the Weyl quantization for observables then we must use the Weyl quantization for dynamical equations. The Weyl quantization of operator (23) leads to superoperator

$$\hat{\mathcal{L}}(\hat{Q}_1,\hat{Q}_2,\hat{P}_1,\hat{P}_2) = \frac{i}{m} \hat{Q}_2^k \hat{P}_1^k - i(m\omega^2 \hat{Q}_1^k + \alpha_{km} \hat{Q}_2^m + \beta_{kms} \hat{Q}_2^m \hat{Q}_2^s) \hat{P}_2^k \ .$$

Let us use the definitions (16,15) of the operators $\hat{Q}_{1,2}$ and $\hat{P}_{1,2}$. The time evolution equation for a quantum observable $\hat{A}_t$ takes the form

$$\frac{d}{dt}\hat{A}_t = \frac{i}{\hbar}[\hat{H},\hat{A}_t] + \frac{i}{\hbar} a_{km} \hat{p}^m \circ [\hat{q}^k, \hat{A}_t] + \frac{i}{\hbar} b_{kms} \hat{p}^m \circ (\hat{p}^s \circ [\hat{q}^k, \hat{A}_t]) \ . \tag{24}$$

Here $\hat{A} \circ \hat{B} = (1/2)(\hat{A}\hat{B} + \hat{B}\hat{A})$ and

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{q}^2}{2} \ .$$

Equation (24) describes quantum analogous of the generalized Lorenz-Rossler-Newton-Leipnik equation (21).

Note that Weyl quantization of $p^m p^s \{p^k, A_t(q,p)\}$ not lead to the term $(-i/\hbar)(\hat{p}^m \circ \hat{p}^s) \circ [\hat{p}^k, \hat{A}_t]$. It gives the term $\hat{p}^m \circ (\hat{p}^s \circ [\hat{q}^k, \hat{A}_t])$ and in general case

$$\hat{p}^m \circ (\hat{p}^s \circ [\hat{q}^k, \hat{A}]) - (\hat{p}^m \circ \hat{p}^s) \circ [\hat{q}^k, \hat{A}] = \frac{1}{4}[\hat{p}^s, [\hat{p}^m, [q^k, \hat{A}]]] \ .$$

7 Fokker-Planck-Type System

Let us consider Liouville operator Λ, which acts on the normed distribution density function $\rho(q,p,t)$ and has the form of second order differential operator

$$\Lambda = d_{qq}\frac{\partial^2}{\partial q^2} + 2d_{qp}\frac{\partial^2}{\partial q \partial p} + d_{pp}\frac{\partial^2}{\partial p^2} + c_{qq} q \frac{\partial}{\partial q} + c_{qp} q \frac{\partial}{\partial p} + c_{pq} p \frac{\partial}{\partial q} + c_{pp} p \frac{\partial}{\partial p} + h \ . \tag{25}$$

Liouville equation

$$\frac{d\rho(q,p,t)}{dt} = \Lambda \rho(q,p,t)$$

with operator (25) is Fokker-Planck-type equation. Weyl quantization of the Liouville operator (25) leads to completely dissipative superoperator $\hat{\Lambda}$, which acts on the matrix density operator and has the form

$$\hat{\Lambda} = -\frac{i}{\hbar}(\hat{H}^l - \hat{H}^r) + \frac{1}{2\hbar}\sum_{j=1,2}\Big((\hat{V}_j^l - \hat{V}_j^r)\hat{V}_j^{*r} - (\hat{V}_j^{*l} - \hat{V}_j^{*r})\hat{V}_j^{*l}\Big) \ .$$

As the result we have the quantum Markovian master equation [28, 30, 35]:

$$\frac{d\hat{\rho}_t}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}_t] + \frac{1}{2\hbar}\sum_{j=1,2}([\hat{V}_j\hat{\rho}_t, \hat{V}_j^*] + [\hat{V}_j, \hat{\rho}_t\hat{V}_j^*]) \ , \tag{26}$$

which is usually called Lindblad equation. Here $\hat{H}$ is Hamilton operator, which has the form

$$\hat{H} = \hat{H}_1 + \hat{H}_2 \ , \quad \hat{H}_1 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2 \ , \quad \hat{H}_2 = \frac{\mu}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}) \ ,$$

where

$$m = -\frac{1}{c_{pq}} \ , \quad \omega^2 = -c_{qp}c_{pq} \ , \quad \lambda = \frac{1}{2}(c_{pp} + c_{qq}) \ , \quad \mu = \frac{1}{2}(c_{pp} - c_{qq}) \ .$$

Operators $\hat{V}_j$ in (26) can be written in the form $\hat{V}_k = a_j \hat{p} + b_j \hat{q}$, where $j = 1, 2$, and complex numbers a_j, b_j satisfy the relations

$$d_{qq} = \frac{\hbar}{2} \sum_{j=1,2} |a_j|^2 \ , \quad d_{pp} = \frac{\hbar}{2} \sum_{j=1,2} |b_j|^2 \ , \quad d_{qp} = -\frac{\hbar}{2} Re(\sum_{j=1,2} a^*_j b_j) \ , \quad \lambda = -Im(\sum_{j=1,2} a^*_j b_j) \ .$$

If $h = -2(c_{pp} + c_{qq})$, then quantum Markovian equation (26) becomes [35]:

$$\frac{d\hat{\rho}_t}{dt} = -\frac{i}{\hbar}[\hat{H}_1, \hat{\rho}_t] + \frac{i(\lambda - \mu)}{\hbar}[\hat{p}, \hat{q} \circ \hat{\rho}_t] - \frac{i(\lambda + \mu)}{\hbar}[\hat{q}, \hat{p} \circ \hat{\rho}_t] -$$

$$-\frac{d_{pp}}{\hbar^2}[\hat{q}, [\hat{q}, \hat{\rho}_t]] - \frac{d_{qq}}{\hbar^2}[\hat{p}, [\hat{p}, \hat{\rho}_t]] + \frac{2d_{pq}}{\hbar^2}[\hat{p}, [\hat{q}, \hat{\rho}_t]] \ .$$

Here d_{pp}, d_{qq}, d_{pq} are quantum diffusion coefficients and λ is a friction constant.

Quantum Fokker-Planck-type system is a quantum dissipative system [18]. The evolution of a pure state $\hat{\rho} = |\Psi><\Psi|$ of this system can be described by nonlinear Schroedinger equation

$$\frac{d}{dt}|\Psi> = \mathcal{H}(\hat{q}, \hat{p}, \Psi)|\Psi> \ ,$$

with Hamilton operator

$$\mathcal{H}(\hat{q}, \hat{p}, \Psi) = \hat{H} + \frac{\mu}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) + \lambda(\hat{q} < \hat{p} > - \hat{p} < \hat{q} >) +$$

$$+\frac{i}{\hbar}\Big(\lambda \hbar^2 - d_{qq} Z(\hat{p}, \hat{p}) - d_{pp} Z(\hat{q}, \hat{q}) + 2d_{pq} Z(\hat{q}, \hat{p})\Big) \ .$$

Here we use the following notions $< \hat{A} > = Sp[\hat{\rho}_t \hat{A}] < \Psi|\hat{A}|\Psi >$ and

$$Z(\hat{A}, \hat{B}) = (\hat{A} - < \hat{A} >) \circ (\hat{B} - < \hat{B} >) + < \hat{A} \circ \hat{B} > - < \hat{A} >< \hat{B} > \ .$$

Note that the quantum diffusion coefficients d_{pp}, d_{qq}, d_{pq} and the friction constant λ are satisfied [35] the following fundamental restrictions:

$$d_{pp} > 0 \ , \quad d_{qq} > 0 \ , \quad d_{pp} d_{qq} - {d_{pq}}^2 \geq \lambda^2 \hbar^2 / 4 \ .$$

8 Lorenz-Type System

Let us consider the evolution of a classical observable $A_t(q, p)$ of dissipative Lorenz-type system

$$\frac{d}{dt} A_t(q, p) = (-\sigma q_1 + \sigma p_1) \frac{\partial A_t(q, p)}{\partial q_1} + (r q_1 - p_1 - q_1 p_2) \frac{\partial A_t(q, p)}{\partial p_1} +$$

$$+(\sigma p_2) \frac{\partial A_t(q, p)}{\partial q_2} + (-b p_2 + q_1 p_1) \frac{\partial A_t(q, p)}{\partial p_2} \ . \qquad (27)$$

This equation for observables $x = q_1$, $y = p_1$ and $z = p_2$, describes classical Lorenz model [32, 36]

$$\frac{dx_t}{dt} = -\sigma x_t + \sigma y_t \ , \quad \frac{dy_t}{dt} = r x_t - y_t - x_t z_t \ , \quad \frac{dz_t}{dt} = -b z_t + x_t y_t \ .$$

The Lorenz model [32] is one of the most famous classical dissipative systems. This system is described by nonlinear differential equations without stochastic terms, but the system demonstrates chaotic behaviour and has strange attractor for $\sigma = 10, \ r = 28, \ b = 8/3$.

The Weyl dynamical quantization of the Lozenz-type equation leads to the quantum Lorenz-type equation

$$\frac{d}{dt}\hat{A}_t = \frac{i}{\hbar}[\frac{\sigma(\hat{p}_1^2 + \hat{p}_2^2)}{2} - \frac{r\hat{q}_1^2}{2}, \hat{A}_t] + \frac{i\sigma}{\hbar}\hat{q}_1 \circ [\hat{p}_1, \hat{A}_t] - \frac{i}{\hbar}\hat{p}_1 \circ [\hat{q}_1, \hat{A}_t] + \frac{i}{\hbar} b \hat{p}_2 \circ [\hat{q}_2, \hat{A}_t] -$$

$$-\frac{i}{\hbar}\hat{q}_1 \circ (\hat{p}_2 \circ [\hat{q}_1, \hat{A}_t]) - \frac{i}{\hbar}\hat{q}_1 \circ (\hat{p}_1 \circ [\hat{q}_2, \hat{A}_t]) \ .$$

Note that Weyl quantization of the term $q_k p_l \{A(q, p), q_m\}$ leads to the term $(i/\hbar)\hat{q}_k \circ (\hat{p}_l \circ [\hat{q}_m, \hat{A}])$, which is equal to $(i/\hbar)\hat{p}_l \circ (\hat{q}_k \circ [\hat{q}_m, \hat{A}])$. It is easy to see from the relation (9) that these terms are not equal to $(i/\hbar)(\hat{q}_k \circ \hat{p}_l) \circ [\hat{q}_m, \hat{A}]$.

9 Conclusions

Dymanical quantization procedure allows to derive dynamical superoperator from dynamical operator and allows to realize a quantization of classical evolution equations. Quantization of a dynamical operator which can be represented by Poisson bracket with the Hamilton function, is defined by the usual canonical quantization. Quantization of a general dynamical operator for non-Hamiltonian system cannot be described by usual canonical quantization procedure. We suggest the quantization scheme which allows to derive quantum analog for the classical non-Hamiltonian systems. The relations (18) and (19) map the operator $\mathcal{L}(q,p,\partial_q,\partial_p)$ which acts on the functions $A(q,p)$ to the superoperator $\hat{\mathcal{L}}$, which acts on the elements $\hat{A}(\hat{q},\hat{p})$ of operator space. If the operator $\mathcal{L}$ is an operator of multiplication on the function $A(q,p) = \mathcal{L}1$, then formula (19) defines the Weyl quantization of the function $A(q,p)$ by the relation $\hat{A} = \hat{\mathcal{L}}I$. Therefore the usual Weyl quantization procedure is a specific case of suggested generalization of Weyl quantization. The suggested approach allows to derive quantum analogs of chaotic dissipative systems with strange attractors.

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