Gravitational Fields with Yang-Mills Curvature

A.N.Tsyrulev

Tver state university, Department of mathematics
Tver, Russia
e-mail: Alexander.Tsyrulev@tversu.ru

Abstract

The $O(1,3)$ gauge invariant equations for the curvature of gravitation compatible with general relativity are obtained in the usual Yang-Mills form. The case of arbitrary nongravitational Yang-Mills gauge field as a source of gravity in the equations is discussed.

The unification of the general relativity and the gauge theory is supposed to be a step in the direction of quantum gravity. There are various versions of the gauge approach to gravity in the field theory [1, 2, 3, 4, 5]. It is the aim of this note to discuss the direct geometrical formulation of the Yang-Mills equations for the Levi-Civita connection curvature, which are compatible with general relativity. We start from the Einstein equations and the Bianchi identity and use the canonical decomposition of the curvature to obtain the corresponding Yang-Mills equations with the Lorentz gauge group. The group can be change over with the isomorphic gauge group $SO(1,C)$ whose Yang-Mills equations are often more convenient in applications. It is essential that sources of gravity are included in our scheme. For example, we discuss shortly arbitrary nongravitational Yang-Mills gauge field as a source of gravity. Among the tools used in this note are the vector bundles and the exterior covariant differentiation, which seem to be more useful in the general relativity than the principal fibre bundles technique.

Let M be a space-time, that is a four-dimensional oriented pseudo-Riemannian manifold, a group Lie G be the structure group of some vector bundle E over M, D a connection(covariant derivative) in E, and F its curvature 2-form on M with values in the Lie algebra. Given the basic representations of G in the fibres, one can identify F with a section of the bundle $E \otimes E^*$. Then the Yang-Mills equations can be written in the form

$$d^D F = 0\,, \qquad \delta^D F = J\,, \tag{1}$$

where d^D is the exterior covariant differential, $\delta^D = -*d^D*$ codifferential, $*$ the Hodge star, J a current of a charge field being, in our context, a section of E.

In Einstein theory of gravitation the mathematical equivalent of a gravitational field is the Riemannian curvature R that is a section of $\Lambda^2T^*M \otimes \Lambda^2T^*M$ with the usual algebraic symmetries. The Hodge star operator naturally acts on Λ^2T^*M and defines the left and right actions $R \mapsto *R$, $R \mapsto R*$, and also the involutive action $R \mapsto *R*$ on the space of curvatures. With respect to the latter action R can be decomposed into a sum [6, 7]

$$R = C + H \tag{2}$$

of two irreducible addends satisfying the duality conditions

$$*C = C*\,, \qquad *H = -H*\,.$$

The decomposition is evidently $O(1,3)$ invariant and the duality conditions are invariant under the proper transformations. Therefore, isolating traceless parts in (2) yields [7]

$$C = W + U, \quad U = \frac{s}{24} g \owedge g, \quad H = \frac{1}{2} g \owedge r_0,$$

where W and r are the Weyl and Ricci tensor fields, g the metric, $r_0 = r - \frac{1}{4}sg$ the traceless part of r, s the scalar curvature, and $\owedge$ the Kulkarni-Nomizu product in S^2T^*M defined by

$$(p \owedge q)_{ijkl} = p_{ik}q_{jl} + p_{jl}q_{ik} - p_{il}q_{jk} - p_{jk}q_{il}\,.$$

We consider R as a 2-form on M with values in the bundle Λ^2T^*M of real-valued 2-forms on M. Any fibre of Λ^2T^*M has natural structure of a Lie algebra with the Lie product $[\alpha, \beta] = \frac{1}{2}(\alpha \cdot \beta - \beta \cdot \alpha)$, which obviously isomorphic to $so(1,3)$. Thus R is like any other gauge field as a section of $\Lambda^2T^*M\otimes\Omega$, where a Lie algebras bundle Ω is Λ^2T^*M (that is a second factor) in our case.

Let ∇ be a Levi-Civita connection on M. Then the differential Bianchi identity $d^\nabla R=0$ holds. On the other hand, ∇_X and the Hodge star on 2-forms commute for any smooth vector field X, that leads to the equalities

$$d^\nabla * R = d^\nabla(*C + *H) = d^\nabla(C * + H * - 2H*) = d^\nabla R * - 2d^\nabla H* = -2d^\nabla H * \,.$$

From the Einstein equations $r-\frac{1}{2}sg=T$, it follows that $s=-tr\,T$, $r_0=T_0=T-\frac{1}{4}(tr\,T)g$ and, consequently, $H=\frac{1}{2}g\text{Ⓐ}T_0$.

For bundles of the kind $\Lambda^2T^*M\otimes\Omega$, where Ω is some vector bundle with a binary operation $\times$, one can define the product $\Lambda^\times$ that is given on the decomposing elements by

$$(\alpha\otimes u)\Lambda^\times(\beta\otimes v) \;=\; \alpha\wedge\beta\otimes(u\times v)\,.$$

It can be verified that $d^\nabla H=-\frac{1}{2}g\wedge^\wedge d^\nabla T_0$ if g and T_0 are considered as T^*M-valued 1-forms on M.

Thus we have the Yang-Mills equations for the Levi Civita connection curvature in the usual form

$$d^\nabla R=0\,,\qquad \delta^\nabla R=\Theta\,, \tag{3}$$

where $\Theta=-*(g\wedge^\wedge d^\nabla T_0)*$ is a section of $T^*M\otimes\Lambda^2T^*M$ and T_0 the traceless part of an energy-momentum tensor. The Hodge star defines a complex structure in Λ^2T^*M. For any section S of a bundle $\Lambda^2T^*M\otimes\Omega$ let be $S'=S-i*S$, $S''=S+i*S$. Then equations (1) can be written as $*d^DF'=iJ$ and (3) as $*d^\nabla R'=i\Theta$. In the latter case the real and imaginary parts give exactly equations (3), however, their meaning in that R' and $\Theta=\frac{1}{2}(\Theta'+\Theta'')$ have to be considered as a $so(3,C)$-valued form on M.

Any solution to the Einstein equations is at the same time a solution to (3) and converse together with some reasonable assumptions is also true. The right side in equations (3) or in their complex analog contains a Lorentz metric and the traceless part of an energy-momentum tensor. Consequently, any interpretation of the equations should not take the metric as a dynamical variable. It is natural to suppose the existence of a primordial principal bundle of orthonormal tetrads(inertial frames) and a connection without torsion in an associated four-vectors bundle. The independent variables should be components ω_{ij} of a Λ^2T^*M-valued connections 1-form and a smooth tetrad fields (e_0,e_1,e_2,e_3) obeying the constraints $[e_i,e_j]=g^{km}(\omega_{kj}(e_i)-\omega_{kj}(e_j))e_m$. The equations (3) are $O(1,3)$ gauge invariant, however, so far we have no suitable variational formulation.

Our purpose is the investigation of links between the Yang-Mills and Einstein theories rather than the search for special solutions of equations (3). However, it is interesting to see some details of the approach for a specific situation. As an example we consider a pure Yang-Mills field F governed by equation (1) with some simple group. The energy-momentum tensor (in the geometrical set of units) of the field is

$$T \;=\; -F\,\cdot^\times\,F\;-\;*F\,\cdot^\times\,*F\,,$$

where the dot means the contraction in Λ^2T^*M and $\times$ the Killing inner product in the Lie algebra. So we get the following expression for the material part of the curvature tensor:

$$H \;=\; -F\,\otimes^\times\,F\;-\;*F\,\otimes^\times\,*F\,.$$

The Ricci contraction of H is equal to T and its total contraction vanishes. The duality condition $*H=H*$ is obviously true. For the complex formulation one can write

$$T \;=\; -F'\,\cdot^\times\,F''\,,\qquad\qquad H \;=\; -\frac{1}{2}\{F'\otimes^\times F''+F''\otimes^\times F'\}\,. \tag{4}$$

Let $\{\alpha^1,\alpha^2,\alpha^3,*\alpha^1,*\alpha^2,*\alpha^3\}$ be a standard orthonormal basis in Λ^2T^*M given by

$$\alpha^1=e^0\wedge e^1,\qquad \alpha^2=e^0\wedge e^2,\qquad \alpha^3=e^0\wedge e^3,$$

$$*\alpha^1=e^3\wedge e^2,\qquad *\alpha^2=e^1\wedge e^3,\qquad *\alpha^3=e^2\wedge e^1,$$

and $\{\epsilon^k = \alpha^k - i * \alpha^k\}$ its complexification. As it was mentioned before, the complex realization of the Yang-Mills field is more convenient in the context of gravitation than the usual one by equation (1). In the basis one has

$$F' = \epsilon^1 \otimes \mathcal{F}_1 + \epsilon^2 \otimes \mathcal{F}_2 + \epsilon^3 \otimes \mathcal{F}_3 ,$$
$$iD_0\mathcal{F}_k = D_l\mathcal{F}_m - D_m\mathcal{F}_l , \quad D_1\mathcal{F}_1 + D_2\mathcal{F}_2 + D_3\mathcal{F}_3 , \tag{5}$$

where D_k denotes covariant derivatives D_{e_k} in the vector bundle, F_k takes values in the complexified Lie algebra (that is $sl(2, C)$ for $su(2)$, for instance), the ordered set (k, l, m) in the first equation (5) is any cyclic transposition of (1,2,3).

In some cases with high symmetry, those make up the most of known solutions to the Einstein equations, the derivative $d\omega_{ij}$ can be explicitly obtained as functions of ω_{ij} and their partial derivatives with respect to coordinates. Then the Bianchi identity is an identity in the direct sense and all information is contained in the decomposition $R' = C' + H'$, expressions (4) and equations (5). Let us, for instance, consider the static space-time with space spherical symmetry and topology $P \times S^2$, where P is a two-dimensional oriented manifold. It can be shown that the general form of metric is

$$ds^2 = Ad\tau^2 - A^{-1}dr^2 - B^2(d\theta^2 + sin^2\theta\, d\phi^2) ,$$

where A and B are positive functions of r only. For the orthonormal basis

$$e^0 = A^{1/2}, \quad e^1 = A^{-1/2}, \quad e^2 = Bd\theta, e^3 = B\, sin\theta\, d\phi ,$$

we have the decomposition mentioned above in the form

$$R' = C' + H' = C^{(1)}\, \epsilon^1 \otimes \epsilon^1 + C^{(2)}(\epsilon^2 \otimes \epsilon^2 + \epsilon^3 \otimes \epsilon^3) + H^{(1)}\, \epsilon^1 \otimes \epsilon^1 + H^{(2)}(\epsilon^2 \otimes \epsilon^2 + \epsilon^3 \otimes \epsilon^3) ,$$

where

$$C^{(1)} = \frac{1}{2}\left(-\frac{1}{2}A'' - A\left(\frac{B'}{B}\right)^2 + \frac{1}{B^2}\right) , \qquad C^{(2)} = -\frac{1}{2}\left(\left(A\frac{B'}{B}\right)' + A\left(\frac{B'}{B}\right)^2\right) ,$$
$$H^{(1)} = \frac{1}{2}\left(-\frac{1}{2}A'' + A\left(\frac{B'}{B}\right)^2 - \frac{1}{B^2}\right) , \qquad HC^{(2)} = \frac{1}{2}\left(A\frac{B''}{B}\right) ,$$

and prime denotes the derivative with respect to r. Here the real part of C' is the Weyl tensor, so we have the equation $C^{(1)} + 2C^{(2)} = 0$, which can be written as

$$(AB^2)'' + 2ABB'' - 2 = 0 .$$

This equation together with (5) determines the space-time totally. For example, one can take $B = r$ and obtain the known Reissner-Nordstrem solution $A = 1 - r_0/r + Q/r^2$, where Q is found from (5) with $\mathcal{F}_2 = \mathcal{F}_3 = 0$ both for the classical electrodynamics and for the Yang-Mills fields.

References

[1] R.Penrose, W.Rindler, Spinors and Spacetime, Cambridge University Press, 1982

[2] D.D.Ivanenko, P.I.Pronin, G.A.Sardanashvily, Gauge Theory of Gravitation, Moscow University Press, 1985(in Russian)

[3] A.Ashtekar, Overview and Outlook, in Canonical Gravity from Classical to Quantum, Springer-Verlag, 1994

[4] N.P.Konopleva, JINR preprint E4-98-207, 1998

[5] B.S.Guilfoyle, B.C.Nolan, LANL gr-qc/9801023

[6] J.Thorp, J.Math.Phys. 10(1969) 1

[7] A.L.Besse, Einstein Manifolds, Springer-Verlag, 1990