How large are extra dimensions?

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Abstract

We discuss Kaluza-Klein theories, in which the fields of the standard model are localized on a 3-brane in a space-time with extra dimensions, and consider the interaction of the localized fields with the "strong" quantum gravity in the bulk. We argue that the interaction of branes with the gravitational field cannot be described within the perturbation theory, and therefore the quantization of gravity in the flat background cannot give correct quantitative results. Thus, the estimates of the size of extra dimensions based on quantum gravity in flat background in the bulk can be only of qualitative character.

The Kaluza-Klein hypothesis has been discussed in theoretical physics for three quarters of a century. In accordance with this hypothesis, space-time may have extra dimensions, which are unobservable for certain reasons. The explanation of this unobservability, which was put forward in the original papers by Kaluza and Klein [1], implies that the extra dimensions are compactified and have a very small size of the order of the Planck's length $l_{Pl} = 1/M_{Pl}$.

The original theory of Kaluza and Klein was pure gravity in space-time $E = M \times S^1$, where M is four-dimensional space-time and S^1 is a circle. If we denote the coordinates in E by $X^N = (x^\nu, y)$, $\quad N = 0, 1, \cdots 4, \quad \nu = 0, 1, \cdots 3, \quad 0 \leq y < R$, the standard gravitational action in this space can be written as

$$S = \frac{1}{16\pi \hat{G}} \int_E \hat{R}\sqrt{-g}\, d^5X, \tag{1}$$

where $\hat{G}$ is the five-dimensional gravitational constant and $\hat{R}$ is the scalar curvature of the metric g_{MN}, $sign\, g = (-, + \cdots +)$. The five-dimensional metric g_{MN} can be decomposed into four-dimensional part and extra components as

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \phi \end{pmatrix}, \quad A_\mu = g_{\mu 4},\, \phi = g_{44}.$$

If g_{MN} does not depend on y, then the scalar curvature R can be expressed as

$$\begin{aligned} \hat{R} &= R_{(4)} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \phi^{-1} - \frac{1}{2} \phi^{-2} \partial_\mu \phi \partial^\mu \phi \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \end{aligned}$$

$R_{(4)}$ being the scalar curvature of the four-dimensional space-time M with metric $g_{\mu\nu}$.

In the original papers by Kaluza and Klein the field ϕ was chosen to be constant and the field A_μ was interpreted as electromagnetic field. Later studies showed that this ansatz was inconsistent, and the reduced theory should also include the scalar field ϕ, which is non-linear. Moreover, the term $\sqrt{-g}$ in the action gives rise to non-linear interaction of the vector field A_μ as well. Thus, having integrated action (1) over the extra dimension, we get an effective four-dimensional action for gravity, interacting with non-linear vector and scalar fields, which cannot be adequately treated by the existing mathematical tools. Nevertheless, this action gives certain relations between multidimensional and four-dimensional coupling constants. In particular, an important point is the relation between the Planck's mass in four dimensions M_{Pl} and the Planck's mass in multidimensional space-time M_B:

$$M_B = \left(\frac{M_{Pl}^2}{R} \right)^{\frac{1}{3}}, \tag{2}$$

It can be easily obtained by putting $A_\mu = 0$, $\phi = 1$ in the ansatz for the five-dimensional metric, which is considered to be its vacuum configuration.

To circumvent the difficulty with non-linearity and non-standard interactions it was proposed to consider the vacuum configuration of the five-dimensional metric and to included gauge, scalar and spinor fields in the multidimensional theory from the very beginning. The inclusion of gauge fields yields extra scalar fields in the reduced theory, because vector field can be decomposed in four-dimensional and extra component as

$$A_M(x,y) = (A_\mu(x,y), A_4(x,y)).$$

Moreover, each component can be expanded in Fourier harmonics in the coordinate y. In the case of a scalar field $\phi(x,y)$ this expansion takes the form

$$\phi(x,y) = R^{-\frac{1}{2}} \sum_n \phi^{(n)}(x) \exp(i\frac{2\pi n y}{R}). \tag{3}$$

The scalar field Lagrangian in 5-dimensional space-time is

$$L = -\frac{1}{2}\partial_M \phi \partial^M \phi - \frac{m^2}{2}\phi^2$$

and the corresponding equation of motion reads

$$(\partial_M \partial^M - m^2)\phi = 0.$$

If we substitute (3) into this 5-dimensional equation of motion, we get that the modes $\phi^{(n)}$ satisfy equations

$$(\partial_\mu \partial^\mu - m_n^2)\phi^{(n)} = 0, \quad m_n^2 = m^2 + \frac{4\pi^2 n^2}{R^2}.$$

The characteristic size R of the extra dimension being of the order of the Planck's length, within the framework of the mathematical apparatus of the modern quantum field theory Kaluza-Klein hypothesis leads to the conclusion that the observed fields (i.e. the fields of the standard model) should be the so called zero modes, i.e. should not depend on the coordinates of extra dimensions. Moreover, for each observed field there should exist a tower of fields with masses of the order of the Planck's mass $M_{Pl} \sim 1/l_{Pl}$, which cannot be observed at the energies, which are available for experiments nowadays.

In 1983 Rubakov and Shaposhnikov put forward a new scenario for Kaluza-Klein theories, which was based on the idea of localization of states [2]. They considered a real Higgs field in 5-dimensional space-time with the Lagrangian

$$L = -\frac{1}{2}\partial_M \phi \partial^M \phi - \frac{\lambda}{4}(\phi^2 - \frac{m^2}{\lambda})^2.$$

The Higgs vacua of the system are $\phi^0 = \pm\frac{m}{\sqrt{\lambda}}$ and an exact solution, connecting these vacua is

$$\phi^{cl}(y) = \frac{m}{\sqrt{\lambda}} th(\frac{my}{\sqrt{2}}).$$

The energy density of the solution is localized in the vicinity of the 3-dimensional hypersurface $y = 0$ and has width of the order $1/m$. This object is called thick domain wall.

The solution ϕ^{cl} can be taken as the vacuum of the theory, and one can study perturbation theory for the variation $u = \phi - \phi^{cl}$. The linearized equation for u in the background ϕ^{cl} is

$$\partial_M \phi \partial^M u + m^2 u - 3(\phi^{cl})^2 u = 0.$$

Since the solution ϕ^{cl} is translation invariant in the four-dimensional space-time, it is natural to look for solutions to this equation in the form

$$u(x,y) = e^{ipx} u(y),$$

where $u(y)$ satisfies the equation

$$[-\frac{d^2}{dy^2} - m^2 + 3(\phi^{cl})^2]u = -p^2 u.$$

This is one-dimensional Schrödinger equation for the wave function u in the potential

$$V(y) = 3m^2 th^2(\frac{my}{\sqrt{2}}) - m^2.$$

This equation has three types of solutions

1. $u_0(y) = \frac{d\phi^{cl}}{dy}$, $p^2 = 0$, $u_0(x, y) = e^{ipx}\frac{d\phi^{cl}}{dy}$
2. $u_1(y)$ is the normalizable solution with the eigenvalue $p^2 = -\frac{3}{2}m^2$, $u_1(x, y) = e^{ipx}u_1(y)$
3. unlocalized solutions $u(x, y)|_{|y|\to\infty} \sim \exp(ipx + ip^4 y)$ with $p^2 = -(3m^2 + (p^4)^2$.

We see that the excitations of the field u, corresponding to $u_0(y)$ and $u_1(y)$, are localized on the domain wall. Moreover, interaction of these modes cannot produce unlocalized modes, if the center of mass energy is less than $\sqrt{6}m$. Thus, for sufficiently small energies the theory describes a 4-dimensional world, although the whole world is, in fact, 5-dimensional.

This model suggests a possibility for extra dimensions to be unobservable, if the fields of the standard model are localized by a certain mechanism on a three-dimensional hypersurface in multidimensional space-time. Rubakov and Shaposhnikov have also proposed the following ansatz for multidimensional metric, which is compatible with this hypothesis [3]:

$$ds^2 = e^{2\sigma(y^i)} g_{\mu\nu} dx^\mu dx^\nu + g_{ik} dx^i dx^k, \tag{4}$$

$\{y^i\}$ being the coordinates of extra dimensions.

In the last years there appeared indications that this scenario can be implemented in the theory of superstrings. In this approach the three-dimensional hypersurfaces are called membranes, or just branes. It is clear that within the framework of this scenario extra dimensions may be much larger, than the Planck's length [4] (or even infinite). In this case the relation (2), which for d extra dimensions looks like

$$M_B \sim \left(\frac{M_{Pl}^2}{R^d}\right)^{\frac{1}{d+2}},$$

yields that the Planck's mass in multidimensional space (in the bulk) M_B may be much less than the Planck's mass in four dimensions M_{Pl}. In other words, gravity in multidimensional space-time becomes "strong" not at the energies of the order of $10^{19}\,GeV$, but at much lower energies, maybe of the order of $10\,TeV$. For example, if we take $M_B \sim 1\,TeV$, the characteristic size of extra dimensions is

- $R \sim 1mm$ for $d = 2$
- $R \sim 5 \cdot 10^{-7}cm$ for $d = 3$
- $R \sim 10^{-9}cm$ for $d = 4$

The effects due to interaction of this "strong" quantum gravity with the fields of the standard model can be observable already at the energies, which are available at the existing colliders.

One of the attempts to calculate such effects was made in paper [5]. The authors considered linearized quantum gravity in the flat background in space-time $E = M \times T^d$, T^d being a d-dimensional torus, interacting with the fields of the standard model localized on a brane. The assumption that the multidimensional background metric can be taken to be flat means that the proper gravitational field of the brane can be neglected. This allowed the authors to use the standard theory of small fluctuations in flat background.

They represented the multidimensional metric in the form

$$g_{MN} = \eta_{MN} + \hat{\kappa}\hat{h}_{MN},$$

where the indices $M, N = 0, 1, \cdots d + 3$, $\eta_{MN} = diag(-1, 1, \cdots 1)$ is the flat Minkowski metric and $\hat{\kappa}^2 = 16\pi\hat{G}$. Substituting the metric g_{MN} into the standard gravitational action (1), rewritten for the case of d extra dimensions, and keeping the terms of the order zero in κ, one gets the so called Fierz-Pauli Lagrangian

$$L = -\frac{1}{4}\left(\partial^M \hat{h}^{NR}\partial_M \hat{h}_{NR} - \partial^M \hat{h}\partial^M \hat{h} - 2\partial_N \hat{h}^{NM}\partial^R \hat{h}^{MR} + 2\partial_N \hat{h}^{NM}\partial_M \hat{h}\right),\ \hat{h} = \hat{h}^M_M. \tag{5}$$

This Lagrangian is invariant under infinitesimal coordinate transformations, which are gauge transformations of the field $\hat{h}_{MN}$. They are parameterized by arbitrary functions ζ_M and have the form

$$\delta \hat{h}^{MN} = \partial_M \zeta_N + \partial_N \zeta_M.$$

Similar to the case of 4-dimensional linearized gravity, these gauge transformations allow one to impose the de Donder-Fock gauge condition, which reads

$$\partial^M(\hat{h}_{MN} - \frac{1}{2}\eta_{MN}\hat{h}) = 0.$$

There remains gauge invariance under the transformations satisfying $\partial_M \partial^M \zeta_N = 0$. This residual gauge transformations, together with the equations of motion in de Donder-Fock gauge

$$\partial^R \partial_R(\hat{h}_{MN} - \frac{1}{2}\eta_{MN}\hat{h}) = 0,$$

imply that the field $\hat{h}_{MN}$ has $(d+2)(d+3)/2-1$ physical degrees of freedom. The problem is to rearrange these degrees of freedom so that they correspond to four-dimensional fields.

A first step is a decomposition in components

$$\hat{h}_{MN} = R^{-\frac{d}{2}}\begin{pmatrix} h_{\mu\nu} + \eta_{\mu\nu}\phi & A_{\mu i} \\ A_{\nu j} & 2\phi_{ij} \end{pmatrix}, \tag{6}$$

where $i, j = 4, 5, \cdots, d+3$, $\phi = \phi^i_i$. Then one has to pass to Kaluza-Klein modes, which are now defined as

$$\begin{aligned} h_{\mu\nu}(x,y) &= \sum_{\vec{n}} h^{\vec{n}}_{\mu\nu}(x)\exp(i\frac{2\pi\vec{n}\vec{y}}{R}) \\ A_{\mu i}(x,y) &= \sum_{\vec{n}} A^{\vec{n}}_{\mu i}(x)\exp(i\frac{2\pi\vec{n}\vec{y}}{R}) \\ \phi_{ij}(x,y) &= \sum_{\vec{n}} \phi^{\vec{n}}_{ij}(x)\exp(i\frac{2\pi\vec{n}\vec{y}}{R}), \end{aligned} \tag{7}$$

where $\vec{n} = (n_1, \cdots, n_d)$ is a vector with integer components, and $\vec{y} = (y_1, \cdots, y_d)$.

All these Kaluza-Klein modes satisfy the equations

$$(\partial_\mu \partial^\mu - m^2_{\vec{n}})f^{\vec{n}} = 0$$

with $m^2_{\vec{n}} = \frac{4\pi^2\vec{n}^2}{R^2}$. Substituting these decompositions into the Fierz-Pauli Lagrangian (5), we get a sum of Lagrangians of each mode $\vec{n}$. In particular, the zero mode Lagrangian turns out to be diagonal in the fields $h^{\vec{0}}_{\mu\nu}, A^{\vec{0}}_{\mu i}, \phi^{\vec{0}}_{ij}$ and looks as follows:

$$\begin{aligned} L^{\vec{0}} &= -\frac{1}{4}(\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} - \partial^\mu h\partial^\mu h - 2\partial_\nu h^{\nu\mu}\partial^\rho h^{\mu\rho} + 2\partial_\nu h^{\nu\mu}\partial_\mu h) - \\ &- \sum_i \frac{1}{4}F^{\nu\mu}_i F_{\nu\mu i} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \sum_{ij}\partial_\mu\phi_{ij}\partial^\mu\phi_{ij}, \end{aligned} \tag{8}$$

where $F_{\nu\mu i} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Thus, we see that the zero mode sector includes graviton, d massless $U(1)$ gauge bosons and $d(d+1)/2$ massless scalars.

The situation with massive modes is more complicated, because the Lagrangian is not diagonal in the fields $h^{\vec{n}}_{\mu\nu}, A^{\vec{n}}_{\mu i}, \phi^{\vec{n}}_{ij}$. Nevertheless, there exist linear combinations of these fields, denoted by $\tilde{h}^{\vec{n}}_{\mu\nu}, \tilde{A}^{\vec{n}}_{\mu i}, \tilde{\phi}^{\vec{n}}_{ij}$ and satisfying the conditions $n_i \tilde{A}^{\vec{n}}_{\mu i} = 0$, $n_i \tilde{\phi}^{\vec{n}}_{ij} = 0$, which diagonalize the Lagrangian for massive modes. The Lagrangian for the mode $\vec{n}$ is

$$
\begin{aligned}
L^{\vec{n}} &= -\frac{1}{2}\left(\partial^\mu \tilde{h}^{\nu\rho,\vec{n}} \partial_\mu \tilde{h}^{-\vec{n}}_{\nu\rho} - \partial^\mu \tilde{h}^{\vec{n}} \partial^\mu \tilde{h}^{-\vec{n}} - 2\partial_\nu \tilde{h}^{\nu\mu,\vec{n}} \partial^\rho \tilde{h}^{-\vec{n}}_{\mu\rho} + 2\partial_\nu \tilde{h}^{\nu\mu,\vec{n}} \partial_\mu \tilde{h}^{-\vec{n}} - \right. \\
&- \left. m^2_{\vec{n}} \tilde{h}^{\nu\mu,\vec{n}} \tilde{h}^{-\vec{n}}_{\nu\mu} + m^2_{\vec{n}} \tilde{h}^{\vec{n}} \tilde{h}^{-\vec{n}}\right) - \sum_i \left(\frac{1}{2} \tilde{F}^{\nu\mu,\vec{n}}_i \tilde{F}^{-\vec{n}}_{\nu\mu i} + m^2_{\vec{n}} \tilde{A}^{\mu,\vec{n}} \tilde{A}^{-\vec{n}}_\mu\right) \\
&- \sum_{ij} \left(\partial_\mu \tilde{\phi}^{\vec{n}}_{ij} \partial^\mu \tilde{\phi}^{-\vec{n}}_{ij} + m^2_{\vec{n}} \tilde{\phi}^{\vec{n}}_{ij} \tilde{\phi}^{-\vec{n}}_{ij}\right).
\end{aligned} \tag{9}
$$

Thus, for each $\vec{n}$ there are one massive spin 2 particle, $(d-1)$ massive vector bosons and $d(d-1)/2$ massive scalar bosons.

The coupling of these fields to the fields of the standard model can be easily found from the action of the standard model in gravitational field, which looks like

$$
S_{SM} = \int d^4x \sqrt{-\tilde{g}} L(\tilde{g}, F, V, S), \tag{10}
$$

where $\tilde{g}$ denotes the metric induced on the brane, and (F, V, S) stand for fermion, vector and scalar fields of the standard model. In accordance with the parameterization (6) the metric induced on the brane situated at $\vec{y} = 0$ is given by

$$
\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa(h_{\mu\nu} + \eta_{\mu\nu}\phi),
$$

where κ is now defined by four-dimensional gravitational constant as $\kappa = \sqrt{16\pi G}$. Taking the variation of the action (10) with respect to metric $\tilde{g}_{\mu\nu}$ and retaining the term of the first order in κ, we get the interaction in the form

$$
-\frac{\kappa}{2} \int d^4x \left(h^{\mu\nu} T_{\mu\nu} + \phi T^\mu_\mu\right),
$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the standard model defined as

$$
T_{\mu\nu} = \left(\eta_{\mu\nu} L - 2\frac{\delta L}{\delta g^{\mu\nu}}\right)|_{g^{\mu\nu} = \eta^{\mu\nu}}.
$$

When rewritten in terms of tilded fields, this interaction looks as follows:

$$
-\frac{\kappa}{2} \sum_{\vec{n}} \int d^4x \left(\tilde{h}^{\mu\nu,\vec{n}} T_{\mu\nu} + \sqrt{\frac{2}{3(d+2)}} \tilde{\phi}^{\vec{n}} T^\mu_\mu\right). \tag{11}
$$

We see that vector gravitons drop out of the interaction Lagrangian and scalar gravitons are coupled only through the dilaton.

This interaction Lagrangian is sufficient to formulate Feynman rules for the interaction of quantized gravity with the fields of the standard model. It was done in paper [5]. It turns out that the interaction of the fields of the standard model with quantum gravity either gives rise to processes, which are absent in the standard model, or gives corrections to the known processes. Both possibilities can be, in principle, studied experimentally. An important point is that though the coupling constant $\kappa = \sqrt{16\pi G}$ is very small, coherent summation of the contributions of a very large number of Kaluza-Klein states may produce observable effects in the energy range $\frac{1}{R} < \sqrt{s} < M_B$. Among those are effects on four-fermion interactions, i.e. on the processes $e^+e^- \to l\bar{l}, q\bar{q}$ and $q\bar{q} - \to l\bar{l}, q\bar{q}$, and on the interaction of fermions with vector bosons, i.e. on the processes e^+e^-, $q\bar{q} \to \gamma\gamma$, W^+W^-, ZZ, gg and $\gamma\gamma$, $gg \to l\bar{l}, q\bar{q}$. The processes, which are absent in the standard model, are associated with radiation of gravitons and manifest themselves as processes with missing energy-momentum, which is carried away by gravitons.

Calculations of cross-sections of some processes involving exchange of gravitons were performed in many papers and certain experimental restrictions on the size of extra dimensions were obtained on the basis of these calculations (see contributions to this Proceedings by J. Ellison, T. Lesiak and A. Zylberstejn).

As we have already mentioned, this theory of interaction of quantized multidimensional gravity with the fields of the standard model was based on the assumption that the proper gravitational field of the brane could be neglected. The validity of the results obtained within this approach depends upon whether this approximation is good or not so good.

In paper [6] first exact solutions for branes interacting with gravity in five-dimensional space-time were found, which allow one to estimate the influence of the proper gravitational field of the brane on these results. First of all, it turned out that if extra dimensions are compact, there should exist at least two branes, and the background metric should be essentially non-flat.

The model is similar to the original Kaluza-Klein theory in the sense that it is a gravity theory in 5-dimensional space-time $E = M \times S^1$ with coordinates $X^M = (x^\mu, y)$, the circle S^1 being of circumference $2R$. But unlike the original Kaluza-Klein theory the gravity interacts with two 3-branes. Thus, the action of the system is

$$\begin{aligned} S &= S_E + S_B + S_F \\ S_E &= \frac{1}{16\pi \hat{G}} \int_E (\hat{R} - \Lambda)\sqrt{-g}\, d^5X \\ S_B &= V_B \int \sqrt{-\tilde{g}_B}\,\delta(y) d^5X \\ S_F &= V_F \int \sqrt{-\tilde{g}_F}\,\delta(y-R) d^5X, \end{aligned} \tag{12}$$

where Λ is the cosmological constant in 5-dimensional space-time and the subscripts "B" (bosonic) and "F" (fermionic) distinguish the branes; the meaning of these notations will be explained below. Moreover, the points (x^μ, y) and $(x^\mu, -y)$ of S^1 are identified, thus giving the orbifold S^1/Z_2.

The Einstein equations for the system are

$$\hat{R}_{MN} - \frac{1}{2} g_{MN}(\hat{R} - \Lambda) = 8\pi\hat{G}\left(V_B\sqrt{\frac{\tilde{g}_B}{g}}\tilde{g}^B_{\mu\nu}\delta^\mu_M\delta^\nu_N\delta(y) + V_F\sqrt{\frac{\tilde{g}_F}{g}}\tilde{g}^F_{\mu\nu}\delta^\mu_M\delta^\nu_N\delta(y-R)\right). \tag{13}$$

If the ansatz for the metric (4), which is the case of one extra dimension looks like

$$ds^2 = e^{2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{14}$$

is substituted into these equations, they reduce to be

$$\begin{aligned} \left(\frac{d\sigma}{dy}\right)^2 &= -\frac{\Lambda}{12} \\ 3\frac{d^2\sigma}{dy^2} &= 8\pi\hat{G}\left(V_B\delta(y) + V_F\delta(y-R)\right). \end{aligned}$$

The solution to these equation with the orbifold symmetry is

$$\begin{aligned} \sigma(y) &= -k|y|, \quad \Lambda = -12k^2 \\ V_B &= -V_F = -\frac{3k}{4\pi\hat{G}}, \end{aligned}$$

where k is a new parameter, which corresponds to the inverse effective thickness of the branes.

Two remarks are in order. First, the space E with metric (14) and σ given by this solution is a piece of five-dimensional anti-de Sitter space in horospherical coordinates. Second, we see that the bosonic brane has positive energy density, whereas the fermionic brane has a negative one. This fact explains our notations for the branes: we assume that on one brane the bosonic vacuum with positive energy density and on the other fermionic vacuum with negative energy density (although both are infinite in the present day field theories) are realized.

Now we have to find the relation between the 5-dimensional gravitational constant $\hat{G}$ and the 4-dimensional G, which will give us the relation between the Planck's masses. To this end we have to calculate the curvature of the metric

$$ds^2 = e^{2\sigma(y)} g_{\mu\nu} dx^\mu dx^\nu + dy^2$$

with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$ and integrate it over the extra dimension. This gives the effective action for 4-dimensional gravity

$$S_{eff} = \frac{1}{16\pi\hat{G}} \int_{-R}^{R} dy e^{-2k|y|} \int_M R(g)\sqrt{-g}\, dx. \tag{15}$$

This leads to the following relation between the gravitational constants

$$\hat{G} = \frac{G}{k}\left(1 - e^{-2kr}\right), \tag{16}$$

which implies that the relation (2) is modified and takes the form

$$M_B = \left(\frac{M_{Pl}^2 k}{1 - e^{-2kR}}\right)^{\frac{1}{3}},$$

Obviously, this relation reduces to (2) for $k \to 0$, i.e. when the effective thickness goes to infinity and the very notion of a brane is meaningless.

Thus, interaction of branes with the gravitational field cannot be described within the perturbation theory, and one has to conclude that the quantization of gravity in the flat background can give at most correct qualitative results, but not quantitative ones. To obtain an adequate quantitative description of the interaction of the "strong" gravity with the fields of the standard model, one has to quantize gravity in the linear approximation in the background of the exact solution. This is an extremely difficult problem, which is extensively studied nowadays.

The general variation of the metric (14) has the form $h_{MN}(x, y)$. Thus, similar to the case of flat background, it corresponds to tensor, vector and scalar gravitons. In [8] it was shown that the vector gravitons can be gauged out, and there exists just one massless scalar graviton, called radion. This degree of freedom describes the fluctuations of the branes with respect to each other. Therefore, it remains to study the tensor modes. Some results in this direction were obtained in [6].

The variation of the Einstein equations (13) leads to the following equation for the transverse and traceless part of the variation of the metric:

$$\left[\frac{1}{2}e^{2k|y|}\partial_\mu\partial^\mu + \frac{1}{2}\frac{d^2}{dy^2} + 2k(\delta(y) - \delta(y - R)) - 2k^2\right] b_{\mu\nu}(x, y) = 0. \tag{17}$$

Since the background metric is translation invariant in four-dimensional space, it is natural to look for solutions to this equation in the form

$$b_{\mu\nu}(x, y) = e^{ipx} b_{\mu\nu} f(y), \quad p^2 = -m^2, \quad b_\mu^\mu = 0, \quad p^\mu b_{\mu\nu} = 0,$$

which reduces the equation (17) to the following equation for the function of one variable:

$$\left[\frac{1}{2}e^{2k|y|}m^2 + \frac{1}{2}\frac{d^2}{dy^2} + 2k(\delta(y) - \delta(y - R)) - 2k^2\right] f(y) = 0. \tag{18}$$

This equation can be solved exactly. To this end we first introduce a variable

$$z = sign(y)\left(e^{k|y|} - 1\right)/k$$

and then a variable $\xi = m(|z| + 1/k)$. When we pass in the equation (18) to the variable ξ, it takes the form

$$\left[\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} + 1 - \frac{4}{\xi^2} + \frac{4}{\xi}\left(\delta(\xi - \xi_1) - \delta(\xi - \xi_2)\right)\right] f(\xi) = 0, \tag{19}$$

where $\xi_1 = m/k$ and $\xi_2 = m/k \exp(kR)$. This equation without the δ-function terms is just the Bessel equation, the general solution being a linear combination of Bessel and Neumann functions $J_2(\xi)$ and $Y_2(\xi)$:

$$Z_2(\xi) = aJ_2(\xi) + bY_2(\xi).$$

The term with δ-functions can be taken into account by imposing the boundary condition

$$Z_2'(\xi) + \frac{2}{\xi} Z_2(\xi) = 0$$

at $\xi = \xi_1 = m/k$ and $\xi = \xi_2 = m/k \exp(kR)$. The first boundary condition can be satisfied by an appropriate choice of the coefficients a and b:

$$Z_2(\xi) = N(m) \left(Y_2 \left(\frac{m}{k} \right) J_2(\xi) - J_2 \left(\frac{m}{k} \right) Y_2(\xi) \right),$$

$N(m)$ being a normalization constant. The second boundary condition defines the mass spectrum of the theory and can be rewritten as

$$J_1(ax)Y_1(bx) - Y_1(bx)J_1(ax) = 0,$$

where $x = mR$, $a = \exp(kR)/kR$ and $b = 1/kR$. There exists a theorem about such combinations of products of Bessel and Neumann functions [7], which asserts that for $a, b > 0$ this combination is an even function of x and its zeros are real and simple. Thus, the non-zero masses of tensor gravitons are defined by the positive zeros γ_n, $n = 1, 2, \cdots$ of this combination and explicitly given by the formula $m_n = \gamma_n / R$.

Let $\{\psi_n(y), n = 0, 1, \cdots\}$ denote a complete orthonormal system of solutions of equation (18), satisfying

$$\int_{-R}^{R} e^{2k|y|} \psi_m(y) \psi_n(y) dy = \delta_{mn}.$$

The functions $\psi_n(y)$ corresponding to the mass eigenvalue $m_0 = 0$ and non-zero masses $m_n = \gamma_n / R > 0$ are given by

$$\begin{aligned} \psi_0(y) &= N_0 \, e^{-2k|y|}, \quad N_0 = \frac{k^{\frac{1}{2}}}{(1 - \exp(-2kR))^{\frac{1}{2}}} \\ \psi_n(y) &= N_n \left(Y_2 \left(\frac{m_n}{k} \right) J_2 \left(\frac{m_n}{k} e^{k|y|} \right) - J_2 \left(\frac{m_n}{k} \right) Y_2 \left(\frac{m_n}{k} e^{k|y|} \right) \right) \end{aligned} \tag{20}$$

We can expand the transverse and traceless part $b_{\mu\nu}(x, y)$ of the variation in this system

$$b_{\mu\nu}(x, y) = \sum_n b^n_{\mu\nu}(x) \psi_n(y),$$

$b^n_{\mu\nu}(x)$ being ordinary 4-dimensional tensor fields, which can be quantized by the standard procedure.

Thus, we have a massless tensor field, an infinite tower of massive tensor fields, just one massless scalar and no massive scalars, which differs from the case of the flat background metric.

Now we have to find the interaction of these fields with the matter on the branes. The general form of this interaction is

$$\hat{\kappa} \int_B h^{\mu\nu}(x, 0) T^B_{\mu\nu} dx + \hat{\kappa} \int_F h^{\mu\nu}(x, R) T^F_{\mu\nu} \sqrt{-\tilde{g}_F} dx, \tag{21}$$

where $T^B_{\mu\nu}$ and $T^F_{\mu\nu}$ are energy-momentum tensors of the matter on bosonic and fermionic branes respectively:

$$T^{B,F}_{\mu\nu} = \frac{\delta L^{B,F}}{\delta g^{\mu\nu}} - \frac{1}{2} g^{B,F}_{\mu\nu} L^{B,F}.$$

A rigorous study of the Randall-Sundrum model shows (a detailed analysis will be published elsewhere) that the interaction of the graviton fields $b^n_{\mu\nu}(x)$ and the massless radion field ϕ with the matter on the bosonic brane is given by

$$\int_B \left(\kappa \sum_n \frac{\psi_n(0)}{N_0} \, b^n_{\mu\nu}(x) T^{\mu\nu} + \frac{\kappa}{4\sqrt{3} k^{\frac{5}{2}} R N_0} e^{kR} \partial_\mu \partial_\nu \phi \, T^{\mu\nu} \right) dx, \tag{22}$$

whereas the interaction with the matter on the fermionic brane is given by

$$\int_F \left(\kappa \sum_n \frac{\psi_n(R)}{N_0} b^n_{\mu\nu}(x) T^{\mu\nu} + \frac{\kappa(1-2kR)}{4\sqrt{3}k^{\frac{5}{2}} R N_0} e^{kR} \partial_\mu \partial_\nu \phi \, T^{\mu\nu} - \frac{\kappa}{\sqrt{3}k^{\frac{1}{2}} R N_0} e^{kR} \phi \, T^\mu_\mu \right) dx. \tag{23}$$

Here κ is expressed in terms of the 4-dimensional gravitational constant G as $\kappa = \sqrt{16\pi G} = \hat{\kappa} N_0$ (the normalization constant N_0 was defined in (20)), which reflects the relation (16) between the constants. Moreover, the action on the fermionic brane has been transformed to the Minkowski metric on this brane.

Thus, we see that the interaction of quantum gravity with the matter on the branes differs considerably, especially the terms with the radion field. The interaction with spin 2 gravitons being similar to the case of the flat metric, we believe that it is this interaction, which is crucial for physical acceptability of the Randall-Sundrum model and for the estimates of the size of extra dimension in this scheme. Our preliminary studies show that the bosonic brane seems to be more suitable to be taken as the visible brane, which is contrary to the belief proliferated in the literature nowadays. This problem will be discussed separately elsewhere.

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