Vacuum Polarization Near Topological Defects in a Brane World

Yu.V. Grats and A.A. Rossikhin

Department of Theoretical Physics, M.V. Lomonosov Moscow State University,
Moscow, 119899 Russia

Abstract

Vacuum polarization effects of a massless scalar field outside straight cosmic string and in the gravitational field of a global monopole imbedded in the Randall-Sundrum brane world are investigated in some detail. It is shown that nontrivial topology which is due to the angular deficit and the existence of extra dimension modifies considerably the spectrum of vacuum fluctuations near defects. Vacuum expectation value of the energy-momentum tensor $\langle T_\mu^\nu \rangle_{vac}^{ren}$ is calculated. It is shown that at small distances vacuum average of the energy-momentum tensor increases significantly as compared to the analogous results obtained in the four-dimensional Einstein's gravity.

1 Introduction

Now it is well realized that at the sufficiently large energy scales $4D$ Einstein's theory does not describe gravity adequately. This means that alternative approaches, in particular those formulated in a space-time of more than four space-time dimensions, must be considered too.

Until recently, Kaluza-Klein theories with a small size of extra dimensions have been discussed in theoretical physics. At the present time, however, the brane world picture becomes more and more popular. In this approach our physical universe is considered to be a 3-brane embedded into a higher dimensional bulk (for the review of this problem see [1] and references therein). Thus, gravity becomes multidimensional while ordinary matter fields are still propagate in the world with three spatial dimensions.

Here we consider the so-called RS2 model with one infinite extra dimension and matter localized on a single positive tension brane [2]. The main goal of our paper is to demonstrate once more that extra dimensions are not hidden, and there are nontrivial effects of the higher-dimensional gravity on the fields trapped to the brane. Our paper is devoted to the effect of vacuum polarization near topological defects imbedded in the brane. We restrict our consideration by the case of a massless scalar field, because it is obvious enough that for the fields of higher spin the result will be the same up to the numerical coefficient.

The paper is organized as follows. As we intend to use the method of dimensional regularization, in Sec.2, the main results of the paper [3] are generalized to the case of RS2 type world of dimension $p+1+1$. Then we consider the particular cases when p-brane contains a multidimensional generalization of a straight string and a global monopole. In Sec. 3, we calculate the Euclidean Green's function for a massless scalar field on the backgrounds under consideration. In Sec. 4, the method of dimensional regularization

is used to obtain the expression for the renormalized vacuum average of the energy-momentum tensor. In Sec. 5, we summarize our results.

In this paper, we use the metric with the signature $(-+\cdots+)$ and the system of units $\hbar = c = 1$.

2 Gravitational field of a straight cosmic string and a monopole in the RS2 space-time of dimension $p+1+1$

Let us start by the formulation of a linearized gravity on a p-brane imbedded in a bulk of dimension $p+1+1$. In this section we briefly reproduce the main ideas of the paper [3].

The action under consideration reads

$$S = \frac{1}{16\pi G_{n+1}} \int d^n x \int dy \sqrt{-g}(R - 2\Lambda) + \\ + \frac{1}{16\pi G_{n+1}} \int_{\text{brane}} d^n x \sqrt{-\hat{g}} \left(\sigma + 16\pi G_{n+1} \mathcal{L}_{\text{matter}}\right) \ , \qquad (1)$$

where $n = p+1$, g_{ab} $(a, b = 0, 1, \ldots, n-1, y)$ is the metric of the bulk space-time, $\hat{g}_{\mu\nu}$ $(\mu, \nu = 0, 1, \ldots, n-1)$ is the induced metric on the brane, G_{n+1} is the bulk gravitational constant. We shell denote coordinates on the p-brane as x^μ; y is the coordinate transverse to the brane.

In the expression above

$$\Lambda = -\frac{n(n-1)}{2} k^2 \quad \text{and} \quad \sigma = -4(n-1)k \ . \qquad (2)$$

This choice of the constants guarantees the existence of a multidimensional solution of the Randall-Sundrun type in the absence of matter on the brane.

As it was shown in [3], there is a gauge, in which the position of the 3-brane remains fixed at $y = 0$ even when matter is present. Proceeding along the same line, let us split up the bulk metric tensor as

$$(g_{ab}) = \begin{pmatrix} \hat{g}_{\mu\nu} & N_\mu \\ N_\nu & N_\lambda N^\lambda + N^2 \end{pmatrix} \ , \ \left(g^{ab}\right) = \frac{1}{N^2} \begin{pmatrix} N^2 \hat{g}^{\mu\nu} + N^\mu N^\nu & -N^\mu \\ -N^\nu & 1 \end{pmatrix} \ , \qquad (3)$$

where N and N_μ are the lapse function and shift vector, respectively.

For the linearization of the Einstein's equations, we write the induced metric as

$$\hat{g}_{\mu\nu} = e^{-2k|y|} \left(\eta_{\mu\nu} + \gamma_{\mu\nu}\right) \ , \qquad (4)$$

and consider $\gamma_{\mu\nu}$, N_μ and $\phi \equiv N^2 - 1$ to be small perturbations.

In this approximation the gauge conditions for the brane to remain at $y = 0$ have the form

$$N_\mu = -\frac{\text{sgny}}{2nk} \gamma_{,\mu} \ , \quad \phi = -\frac{\text{sgny}}{nk} \gamma_{,y} \ , \qquad (5)$$

$$\tilde{\gamma}^{,\mu}_{\mu\nu} = 0 \qquad (6)$$

where $\gamma = \eta^{\mu\nu}\gamma_{\mu\nu}$ and $\tilde{\gamma}_{\mu\nu} \equiv \gamma_{\mu\nu} - \frac{1}{n}\eta_{\mu\nu}\gamma$ is the traceless part of $\gamma_{\mu\nu}$.

Following [3] we obtain that for $k > 0$ and $q^2 > 0$ the solution for the metric perturbations $\tilde{\gamma}_{\mu\nu}$ on the brane in the momentum space reads

$$\tilde{\gamma}_{\mu\nu}(q, y = 0) = \frac{8\pi G_{n+1}}{|q|} W_{\mu\nu}(q) \frac{\mathrm{K}_{n/2}\left(|q|/k\right)}{\mathrm{K}_{n/2-1}(|q|/k)} \ , \tag{7}$$

where

$$W_{\mu\nu}(q) = t_{\mu\nu}(q) - \frac{1}{n-1}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) t(q) \ , \tag{8}$$

and $t_{\mu\nu}$ is the energy-momentum tensor of a matter localized on the brane, $t = \eta^{\mu\nu}t_{\mu\nu}$.

Corresponding solution for the trace γ has the form

$$\gamma(x, y = 0) = 8\pi G_{n+1} \frac{n}{n-1} k \int \frac{d^n q}{(2\pi)^n} e^{iqx} \frac{t(q)}{q^2} \ . \tag{9}$$

Multidimensional generalization of the energy-momentum tensor of a straight infinitely thin string (object of co-dimension two) has the form

$$t^S_{\mu\nu} = -\mu\delta^2(\vec{x}) \left(\eta_{\mu\nu} - \sum_{i=1}^{2} \delta_{i\mu}\delta_{i\nu}\right) \ , \tag{10}$$

where $\vec{x} = (x^1, x^2)$ and μ is the energy per unit volume of the string which determines its deficit angle.

In this particular case

$$W^S_{\mu\nu}(q) = \mu \prod_{i=3}^{n} \delta(q_i) \frac{(2\pi)^{n-2}}{n-1} \left[(n-1)\sum_{i=1}^{2} \delta_{i\mu}\delta_{i\nu} - \eta_{\mu\nu} - (n-2)\frac{q_\mu q_\nu}{q^2}\right] \ , \tag{11}$$

and

$$t^S(q) = -(n-2)\mu \prod_{i=3}^{n} \delta(q_i)(2\pi)^{n-2} \ . \tag{12}$$

Now we can obtain the expressions for the induced metric on the brane and corresponding intrinsic Ricci tensor and scalar curvature. It can be shown that, in contrast to its four-dimensional equivalent, the space-time of a string on the brane is not Ricci flat, but the linearized intrinsic scalar curvature is still equal to zero everywhere outside the core where it has a δ-like singularity.

Let us now consider the case of a monopole. For the corresponding $n = p + 1$ dimensional energy-momentum tensor we have

$$t^M_{\mu\nu} = -\eta^2 \frac{r_\mu r_\nu}{r^4} - \frac{\eta^2}{r^2}\left(\eta_{\mu\nu} - \sum_{i=1}^{3} \delta_{\mu i}\delta_{\nu i}\right) \ , \tag{13}$$

where $r^\mu = (0, x^1, x^2, x^3, 0, \ldots, 0)$, $r^2 = \eta_{\mu\nu}r^\mu r^\nu$ and η^2 is the energy scale of symmetry braking which determines the deficit solid angle of the monopole.

In this case

$$W^M_{\mu\nu}(q) = \frac{\eta^2\pi^2}{|q|}\prod_{i=4}^{n}\delta(q_i)\frac{(2\pi)^{n-3}}{n-1}\Big[(n-1)\sum_{i=1}^{3}\delta_{i\mu}\delta_{i\nu} - 2\eta_{\mu\nu} - (n-3)\frac{q_\mu q_\nu}{q^2}\Big] \ , \qquad (14)$$

and

$$t^M(q) = -2(n-2)\frac{\eta^2\pi^2}{|q|}\prod_{i=4}^{n}\delta(q_i)(2\pi)^{n-3} \ .$$

In the case of a global monopole the linearized intrinsic Ricci tensor is a sum of contribution from the higher dimensional bulk and a term which, for $n = 4$, coincides with the result obtained in the Einstein's theory. At the same time the scalar curvature

$$\hat{R}^M = 16\pi G_n\frac{\eta^2}{r^2} \qquad (15)$$

is not equal to zero but contains no contribution from the extra dimension.

3 Euclidean Green's function

Euclidean Green's function of a scalar field in a space-time of dimension n is the solution of the Poisson equation

$$\Big(\hat{g}^{\mu\nu}_E\hat{\nabla}_\mu\hat{\nabla}_\nu - \xi\hat{R}\Big)G^E(x,x') = -\frac{\delta^n(x-x')}{\sqrt{\hat{g}_E(x)}} \ , \qquad (16)$$

where $\hat{g}^E_{\mu\nu}$ is obtained from the metric under consideration by the Wick rotation.

Proceeding along the same line as in the paper [4], let us write this equation in the form

$$\Delta^n_0 G^E(x,x') = -\delta^n(x-x') - VG^E(x-x') \ , \qquad (17)$$

where $\Delta^n_0 \equiv \delta^{\mu\nu}\partial_\mu\partial_\nu$ is the Laplace operator in the n-dimensional Euclidean space, and

$$V = \partial_\mu\left(\sqrt{\hat{g}}_E\hat{g}^{\mu\nu}_E\partial_\nu\right) - \delta^{\mu\nu}\partial_\mu\partial_\nu - \xi\hat{R}\sqrt{\hat{g}_E} \ . \qquad (18)$$

When the use of perturbation theory is justified, the solution of the equation (17) can be written as

$$G^E = G^E_0 + G^E_0VG^E_0 + G^E_0VG^E_0VG^E_0 + \ldots , \qquad (19)$$

where G^E_0 is the Green's function of the Poisson equation in the n-dimensional Euclidean space.

Some comments about the case of a straight string. As it was mentioned in the previous section, $\hat{R}$ has a δ-like singularity on the string world sheet, and Eq. (16) becomes ill-defined. The possible way to avoid this problem is to smooth out the singularity [5, 6]. As the result, one finds that when (at the end of calculations) the regularization of the conical singularity is removed, the dependence of the Green's functions on ξ vanishes. So, the term proportional to the scalar curvature can be ruled out of V.

Now using equations (18), (19) and the explicit expression for the function G_0^E, we find that the first-order correction to the Euclidean Green's function $G_1^E = G_0^E V G_0^E$ reads

$$G_1^E = G_{Eins} + G_{bulk} \ , \tag{20}$$

where

$$G_{Eins} = \kappa_n \int \frac{d^n q}{(2\pi)^n} \frac{e^{iqx}}{q^2} \int \frac{d^n p}{(2\pi)^n} \frac{e^{-ip(x-x')}}{p^2 (p-q)^2} \times$$
$$\times \left[2W_{\mu\nu}(q) p^\mu p^\nu - t \frac{p_\mu q^\mu}{n-1} - \xi q^2 \frac{\hat{R}}{\kappa_n} \right] \ , \tag{21}$$

$$\kappa_n \equiv 8\pi G_{n+1} \frac{(n-2)}{2} k \ , \qquad \begin{cases} \hat{R} = 0 & \text{string} \\ \hat{R} = \hat{R}^M & \text{monopole} \end{cases}$$

coincides with the expression obtained for the first-order corrections to the Green's function in the n-dimensional Einstein gravity [4], while

$$G_{bulk} = 8\pi G_{n+1} \int \frac{d^n q}{(2\pi)^n} e^{iqx} \frac{\mathrm{K}_{n/2-2}(|q|/k)}{|q| \mathrm{K}_{n/2-1}(|q|/k)} \int \frac{d^n p}{(2\pi)^n} \frac{e^{-ip(x-x')}}{p^2 (p-q)^2} W_{\mu\nu}(q) p^\mu p^\nu \tag{22}$$

is a contribution from the bulk.

4 Calculation of the vacuum expectation value $\langle T_\mu^\nu \rangle_{vac}^{ren}$

To calculate the vacuum expectation value $\langle T_\mu^\nu \rangle_{vac}^{ren}$, let us start from the formal expression

$$\langle T_\mu^\nu \rangle_{vac} = \lim_{x' \to x} D_\mu^{\nu'}(x, x') G^E(x, x') \ , \tag{23}$$

where $D_\mu^{\nu'}$ is a differential operator whose explicit form is determined by the classical expression for the stress-energy tensor. For a massless scalar field with arbitrary coupling and in the lowest order with respect to the gravitational constant, we have

$$D_\mu^{\nu'} = (1 - 2\xi) \nabla_\mu^{\nu'} - 2\xi \nabla_\mu^\nu + \left(2\xi - \frac{1}{2} \right) \delta_\mu^\nu \nabla_\lambda^{\lambda'} \ . \tag{24}$$

Substituting expressions (20), (21), and (22) into (23), we obtain, that in the lowest order approximation vacuum expectation value of the energy-momentum tensor consists of two parts

$$\langle T_\mu^\nu \rangle_{vac} = \langle T_\mu^\nu \rangle_{Einst} + \langle T_\mu^\nu \rangle_{bulk} \ . \tag{25}$$

Subsequent calculations can be performed along the same line as in the case of cosmic strings in the four-dimensional Einstein's theory.

Integrals over $d^n p$ can be evaluated using the method of dimensional regularization. After the integration over angles in the momentum space one must perform the replacement $n \to D = n - 2\epsilon$ and multiply the regularized vacuum average of the stress-energy tensor by $(\lambda)^{n-D}$ (λ is an arbitrary parameter with the dimension of mass) to restore the dimension of $\langle T_\mu^\nu \rangle_{vac}$. Using the results of paper [4] one can find that

$$\langle T^\nu_\mu\rangle^{reg}_{Einst} = \kappa_D \frac{\lambda^{n-D}}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)\Gamma(D/2)^2}{(D-2)(D+1)\Gamma(D)} \int \frac{d^n q}{(2\pi)^n} e^{iqx} |q|^{D-2} \times$$
$$\times \left\{ 4(D^2-1)(\xi-\xi_c) \left(\delta^\nu_\mu - \frac{q_\mu q^\nu}{q^2}\right) \left[\frac{t(q)}{2(D-1)} + \xi \frac{\hat{R}}{\kappa_D}\right] + W^\nu_\mu(q) \right\} , \qquad (26)$$

with $\xi_c = \frac{1}{4}\frac{D-2}{D-1}$, and

$$\langle T^\nu_\mu\rangle^{reg}_{bulk} = \frac{8\pi G_{n+1}}{4(D+1)(D/2-1)} \Gamma(2-D/2) \frac{\Gamma^2(D/2)}{\Gamma(D)} \times$$
$$\times \frac{1}{(4\pi)^{D/2}} \int \frac{d^n q}{(2\pi)^n} e^{iqx} \frac{\mathrm{K}_{D/2-2}\left(|q|/k\right)}{\mathrm{K}_{D/2-1}(|q|/k)} |q|^n \left(\frac{|q|}{\lambda}\right)^{D-n} W^\nu_\mu(q) \ . \qquad (27)$$

As it is well known [7], in four space-time dimensions the divergent part of $\langle T^\nu_\mu\rangle$ for the massless scalar field reads

$$\langle T^\nu_\mu\rangle_{div} = \frac{\Gamma(2-D/2)}{2(4\pi)^{D/2}} \left(\frac{\lambda_0}{\lambda}\right)^{D-4} \left(\frac{1}{90} H^\nu_\mu - \frac{1}{90} {}^{(2)}H^\nu_\mu + \left(\frac{1}{6}-\xi\right)^2 {}^{(1)}H^\nu_\mu \right) \ . \qquad (28)$$

In the expression above λ_0 is the infrared cutoff which must be introduced in the case of a massless field, and for our metric been taken in the linear approximation we find

$$H_{\mu\nu} = -4\Delta_0 \hat{R}_{\mu\nu} + 2\hat{R}_{;\,\mu\nu} \qquad (29)$$
$${}^{(1)}H_{\mu\nu} = 2\hat{R}_{;\,\mu\nu} - 2\delta_{\mu\nu}\Delta_0 \hat{R} \qquad (30)$$
$${}^{(2)}H_{\mu\nu} = \hat{R}_{;\,\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}\Delta_0 \hat{R} - \Delta_0 \hat{R}_{\mu\nu} \ . \qquad (31)$$

Let us first consider the case of a string. In this particular case all the integrations in Eq. (26) can be performed explicitly, and we obtain that for the physical case of 3-brane ($n = 4$) zero-zero component of $\langle T^\nu_\mu\rangle^{ren}_{Einst}$ is

$$\langle T^0_0\rangle^{ren}_{Einst} = \frac{G_4\mu}{90\pi^2 r^4} + (\xi - \frac{1}{6}) \frac{4G_4\mu}{\pi^2 r^4} \ . \qquad (32)$$

At the same time contribution from the bulk (27) is divergent.

Subtracting (28) from regularized expression (27) at $n = 4$, we get that outside the core of the string

$$\langle T^\nu_\mu\rangle^{ren}_{bulk} = -\frac{G_4\mu}{360\pi} \int \frac{d^2 q}{(2\pi)^2} e^{iqx} \frac{\mathrm{K}_0\left(|q|/k\right)}{|q| k \mathrm{K}_1(|q|/k)} |q|^4 \ln\left(\frac{|q|}{\lambda}\right) \times$$
$$\times \left[3\sum_{i=1}^{2} \delta_{i\mu}\delta^\nu_i - \delta^\nu_\mu - 2\frac{q_\mu q^\nu}{q^2} \right] \ . \qquad (33)$$

¿From Eq (33) we can obtain, that, at large distances from the string contribution from the bulk is negligible compared to (32)

$$\langle T^0_0\rangle^{ren}_{bulk} = \frac{4G_4\mu}{45\pi^2} \frac{1}{k^2 r^6} \left[C_1 - 2\ln(r\lambda)\right] \ , \qquad (34)$$

while near the string, at $r \ll k^{-1}$,

$$\langle T_0^0 \rangle_{bulk}^{ren} = \frac{G_4 \mu}{80\pi^2} \frac{1}{kr^5} \left[C_2 - \ln(r\lambda) \right] \ , \tag{35}$$

where C_1 and C_2 are some numerical constants.

We see, that contrary to the case of the Einstein's gravity the expression for the total vacuum expectation value $\langle T_{\mu\nu} \rangle_{vac}^{S\ ren}$ in the string space-time contains a λ-dependent term which appears due to the renormalization procedure. Moreover, at short distances the bulk contribution exceeds the ordinary one by far.

In the case of monopole $\langle T_\mu^\nu \rangle_{Einst}^{ren}$ can be represented in the form

$$\langle T_\mu^\nu \rangle_{Einst}^{M\ ren} = \frac{G_4 \eta^2}{2\pi r^4} \left[A_\mu^\nu(\xi) + B_\mu^\nu(\xi) \ln \lambda r \right] \ . \tag{36}$$

For more details see [8].

Renormalized contribution from the bulk can be calculated with the use of the same technique as in the case of a string, and has the form

$$\langle T_\mu^\nu \rangle_{bulk}^{ren} = -\frac{\pi G_5 \eta^2}{720} \int \frac{d^3q}{(2\pi)^3} e^{iqx} \frac{\mathrm{K}_0\left(|q|/k\right)}{\mathrm{K}_1(|q|/k)} |q|^2 \ln\left(\frac{|q|}{\lambda}\right) \times \left[3 \sum_{i=1}^{2} \delta_{i\mu} \delta_i^\nu - 2\delta_\mu^\nu - \frac{q_\mu q^\nu}{q^2} \right] \ . \tag{37}$$

¿From the last equation we can obtain, that, at large distances contribution from the bulk is of the order $(kr)^2$ compared to (36). Contrary, at $r \ll k^{-1}$ we have

$$\langle T_0^0 \rangle_{bulk}^{ren} = -\frac{G_4 \eta^2}{480 k r^5} \ . \tag{38}$$

Thus, as in the case of a string $\langle T_\mu^\nu \rangle_{bulk}$ is much larger than (36).

5 Conclusions

In this paper we consider the vacuum polarization effects outside topological defects with angular deficit in the RS2 brane world. It is shown, that in the lowest order of the perturbation theory $\langle T_{\mu\nu} \rangle_{vac}^{ren}$ consists of two parts. The first term depends on the angular deficit only and reproduces the result obtained previously in the $4D$ Einstein's gravity. The second contribution depends on the scale k and arises due to the possibility of gravity to propagate in the five-dimensional bulk. Relative contribution of this term tends to zero as $kr \to \infty$, but it becomes significant at short distances. Another interesting feature is the appearance of an arbitrary mass scale λ in the $\langle T_{\mu\nu} \rangle_{vac}^{ren}$ both in the string and monopole cases. From the first glance the loss of uniqueness must lead to some problems because the ordinary approach to the back reaction problem consists of solving the semiclassical Einstein's equations with $\langle T_{\mu\nu} \rangle_{vac}^{ren}$ in its right-hand side. Analogous question was considered in the case of a global monopole in the four-dimensional Einstein's theory [8]; it was shown that any variation of λ may be absorbed by the coefficients before the fourth-order terms appearing in the left-hand side of the one-loop Einstein's equations.

Acknowledgments

This work was supported by the Russian Foundation for Basic Research, grant 99-02-16132.

References

[1] V. A. Rubakov, hep-ph/0104152

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690

[3] I.Ya. Aref'eva *et al.*, Nucl. Phys. B590 (2000) 273

[4] D. V. Gal'tsov, Yu. V. Grats, and A. A. Lavrent'ev, Phys. Atomic Nucl. 58 (1995) 516

[5] B. Allen and A. C. Ottewill, Phys. Rev. D42 (1990) 2669

[6] B. Allen, J. G. Mc Laughlin and A. C. Ottewill, Phys. Rev. D45 (1992) 4486

[7] T. S. Bunch, Journ. Phys. A12 (1979) 517

[8] F. D. Mazzitelli and C. O. Lousto, Phys. Rev. D43 (1991) 468