Construction of Asymptotic Solutions for the Generalized Hénon-Heiles System in Nonintegrable Case

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Abstract

The Hénon–Heiles system in the general form is studied. In a nonintegrable case new solutions have been found as formal Laurent series, depending on three parameters. One of the parameters determines a location of the singularity point, other parameters determine coefficients of the Laurent series. For some values of these two parameters the obtained Laurent series coincide with the Laurent series of the known exact solutions.

1 The Hénon–Heiles Hamiltonian

Let us consider a three-dimensional galaxy with an axial-symmetric and time-independent potential function. The equations of galactic motion admit two well-known integrals: energy and angular momentum. If we know also the third integral of motion, then we can solve the motion equations by the method of quadratures. However, the third integral as a polynomial function does not exist in the general case.

In the 1960s numerical [1] and asymptotic methods [2, 3] have been developed to show either existence or absence of the third integral for some polynomial potentials. Hénon and Heiles wrote [1]: "In order to have more freedom of experimentation, we forgot momentarily the astronomical origin of the problem and consider its general form: does an axisymmetrical potential admit a third isolating integral of motion ?". They have proposed the following Hamiltonian:

$$H = \frac{1}{2}\Big(x_t^2 + y_t^2 + x^2 + y^2\Big) + x^2y - \frac{1}{3}y^3, \qquad (1)$$

because: **(a)** it is analytically simple; this makes the numerical computations of trajectories easy; **(b)** at the same time, it is sufficiently complicated to give trajectories, which are far from trivial. Indeed, Hénon and Heiles found that for low energies this system appeared to be integrable, in so much as trajectories (numerically integrated) always lay on well-defined two-dimensional surfaces. On the other hand, they also obtained that for high energies many of these integral surfaces were destroyed and that phase space acquired large ergodic regions. The Hénon–Heiles system became a paradigm of chaotic Hamiltonian dynamics.

Let us consider the **Hénon–Heiles** Hamiltonian in the general form:

$$H = \frac{1}{2}\Big(x_t^2 + y_t^2 + \lambda x^2 + y^2\Big) + x^2y - \frac{C}{3}y^3, \qquad (1')$$

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where C and λ are numerical parameters.

Investigations of the generalized Hénon–Heiles system:

$$\left\{\begin{array}{lcl} x_{tt} & = & -\lambda x - 2xy, \\ y_{tt} & = & -y - x^2 + Cy^2, \end{array}\right. \qquad (2)$$

can be separated on the following ways:

1. The solutions of (2) have been investigated in the complex time plane using numerical integration techniques [4–6].

2. The second integral as formal power series in phase variables x, x_t, y and y_t (Gustavson integral) has been constructed for the Hénon–Heiles system both in the original ($\lambda = 1$, $C = 1$) [3] (see also [7]) and in the general forms [8]. To obtain the second integral Gustavson has constructed the normal form of Hamiltonian (1). Using the Bruno algorithm [9] V.F. Edneral has constructed the Poincaré–Dulac normal form and found (provided that all phase variables are small) local families of periodic solutions of Hénon–Heiles system both in the original [10] and in the general [11] forms.

3. The singularities at the fixed points in phase space are locally analyzed via normal form theory, whereas the singularities in the complex (time) plane are studied by the Painlevé analysis. Using this analysis three integrable cases of system (2) have been found:

$$\begin{array}{lll} \text{(i)} & C = -1, & \lambda = 1, \\ \text{(ii)} & C = -6, & \lambda \textbf{ is an arbitrary number}, \\ \text{(iii)} & C = -16, & \lambda = \frac{1}{16}. \end{array}$$

The nontrivial cases are (ii) and (iii), case (i) is separable. To prove the integrability in the cases (ii) and (iii) the second integrals have been found [12–15][2]. Further development of the Painlevé approach has led to finding of the Bäcklund transformations [17] and the Lax representations [18, 19] for the integrable cases of (2).

4. It is known that the variable y as a solution of the Hénon–Heiles system has to satisfy the following fourth-order equation:

$$y_{tttt} = (2C-8)y_{tt}y - (4\lambda+1)y_{tt} + 2(C+1)y_t^2 + \frac{20C}{3}y^3 + (4C\lambda-6)y^2 - \lambda y - 4H. \qquad (3)$$

To find a special solution of this equation one can assume that y satisfies some more simple equation. For example, it is well known that the Hénon–Heiles system and, hence, equation (3) have solutions in terms of the Weierstrass elliptic functions satisfying the first-order differential equation:

$$y_t^2 = \mathcal{A}y^3 + \mathcal{B}y^2 + \mathcal{C}y + \mathcal{D}, \qquad (4)$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are some constants.

E.I. Timoshkova [20] generalized equation (4):

$$y_t^2 = \mathcal{A}y^3 + \mathcal{B}y^2 + \mathcal{C}y + \mathcal{D} + \mathcal{G}y^{5/2} + \mathcal{E}y^{3/2} \qquad (5)$$

and found new one-parameter sets of solutions of the Hénon–Heiles system in nonintegrable cases ($C = -\frac{4}{3}$ or $C = -\frac{16}{5}$, λ is an arbitrary number).

In the present paper I use the Painlevé method to find asymptotic solutions of the Hénon–Heiles system in the case $C = -\frac{16}{5}$.

[2] A geometrical interpretation of the second integral in the case (ii) is presented in [16].

2 The Painlevé property

The Painlevé property for ordinary differential equations (ODE's) is defined as follows. Solutions of a system of ODE's are regarded as analytic functions, may be with isolated singular points [21]. A singular point of a solution is said *critical* if the solution is multivalued in the neighborhood of it and *movable* if its location depends on initial conditions.

Definition. A system of ODE's is said to have ***the Painlevé property*** *if its general solution has no movable critical singularity* [22].

All solutions of such systems (systems of P-type) are expressible as simple Laurent series. As known a Hamiltonian system in a $2s$–dimensional phase space is called *completely integrable* , if it possesses s independent integrals which commute with respect to the associated Poisson bracket. When this is the case, the equations of motion are (in principal, at least) separable and solutions can be obtained by the method of quadratures. Since the work of S.V. Kovalevskaya [23] (see also [24]) on the motion of a rigid body about a fixed point, the Painlevé property has been proposed as a criterion for complete integrability [25–27]. If the system misses the Painlevé property (has complex or irrational "resonances"), then the system cannot be "algebraically integrable" [28] (see also [29] and references there in). N. Ercolani and E.D. Siggia [30] advance arguments as to why the Painlevé test works, *i.e.* they showed how to exploit the singular analysis to yield the integrals. They proved a theorem which demonstrates that the singularity analysis provides bounds on the degrees of polynomial integrals for a large class of separable systems.

3 The Painlevé test

The Painlevé test is any algorithm designed to determine necessary conditions for a differential equation to have the Painlevé property. The original algorithm developed by Painlevé [22] is known as the α-method. The method of S.V. Kovalevskaya [23] is not as general as the α-method but is more simple than it is. In 1980, motivated by the work of S.V. Kovalevskaya, M.J. Ablowitz, A. Ramani and H. Segur [31] developed a new algorithm of the Painlevé test for ODE's[3]. Using this algorithm one can determine whether a system of ODE's admits movable branch points, either algebraic or logarithmic. This algorithm can be used not only to isolate values of parameters of integrable cases, but also to find asymptotic special solutions even in nonintegrable cases.

If one substitutes in a system of ODE's, for example, in system (2):

$$x = \widetilde{cx}_{\alpha}(t-t_0)^{\alpha} + \sum_{j=1}^{N_{max}} \widetilde{cx}_{j+\alpha}(t-t_0)^{j+\alpha} \quad \text{and} \quad y = \widetilde{cy}_{\beta}(t-t_0)^{\beta} + \sum_{j=1}^{N_{max}} \widetilde{cy}_{j+\beta}(t-t_0)^{j+\beta},$$

[3]In [31] Ablowitz *et al.* have proved that if a partial differential equation (PDE) is solvable by the inverse scattering transform and a system of ODE's is obtained from this PDE by the exact similarity reduction, then the solutions (of this system of ODE's), associated with Gel'fand–Levitan–Marchenko equation, will possess the Painlevé property. Furthermore, they conjecture that, when all the ODE's obtained by exact similarity transforms from a given PDE have the Painlevé property, perhaps, after a change of variables, then PDE will be "integrable". A powerful generalization of the Painlevé test for both nonlinear ordinary and partial differential equations was developed by J.Weiss, M.Tabor and G.Carnevale [32, 33] (see also [16,34–40]).

where N_{max}, α, β, $\widetilde{cx}_\alpha$ and $\widetilde{cy}_\beta$ are some known constants, then the system of ODE's transforms into a set of linear algebraic systems in coefficients $\widetilde{cx}_k$ and $\widetilde{cy}_k$. In the general case one can obtain the exact solutions (in the form of formal Laurent series) only if one solves infinity number of systems: $N_{max} = \infty$. On the other hand, if one solves a finite number of systems then one obtains asymptotic solutions. With the help of some computer algebra system, for example, the system **REDUCE** [41, 42], these systems can be solved step by step and asymptotic solutions can be automatically found with any accuracy. But previously one has to determine values of constants α, β, $\widetilde{cx}_\alpha$ and $\widetilde{cy}_\beta$ and to analyze systems with zero determinants. Such systems correspond to new arbitrary constants or have no solutions. The powers at which enter new arbitrary constants are called *resonances*. The Painlevé test gives all information about possible dominant behaviors and resonances.

For the generalized Hénon-Heiles system two possible dominant behaviors and resonance structures have been found [5, 25, 38]:

Case 1:	*Case 2*: $(\beta < \Re e(\alpha) < 0)$
$\alpha = -2$, $\beta = -2$, $\widetilde{cx}_\alpha = \pm 3\sqrt{2+C}$, $\widetilde{cy}_\beta = -3$ $r = -1,\ 6,\ \frac{5}{2} \pm \frac{\sqrt{1-24(1+C)}}{2}$.	$\alpha = \frac{1\pm\sqrt{1-48/C}}{2}$, $\beta = -2$, $\widetilde{cx}_\alpha = a_1$ **(an arbitrary number)**, $\widetilde{cy}_\beta = \frac{6}{C}$, $r = -1,\ 0,\ 6,\ \pm\sqrt{1-\frac{48}{C}}$.

The variable r denotes resonances: $r = -1$ corresponds to arbitrary parameter t_0; $r = 0$ (in the *Case 2*) corresponds to arbitrary parameter a_1; other values of r are numbers of system with zero determinants (if some values are not integer, then the system is not of P-type). It is possible that an original system is not of P-type, but after some change of variables the obtained system is of P-type[4].

At $C = -\frac{16}{5}$ in the *Case 1* some values of r are not rational. This result means, that the Hénon–Heiles system with $C = -\frac{16}{5}$ is a nonintegrable system. There is no algorithm to find the general solution for such system. To find a special asymptotic solutions let us consider the *Case 2*. In this case we obtain $\alpha = -\frac{3}{2}$, so to get single-valued functions we have to introduce new variables $z = x^2$ and to consider the following system instead of (2):

$$\left\{ \begin{array}{lcl} z_{tt}z & = & \frac{1}{2}z_t^2 - 2\lambda z^2 - 4z^2 y, \\ y_{tt} & = & -y - z - \frac{16}{5}y^2. \end{array} \right. \tag{6}$$

It is easy to see that $r = -1,\ 0,\ 4,\ 6$, hence, the systems with zero determinants have numbers 4 and 6. Substituting ($\tilde{a}_1 \equiv a_1^2$)

$$z = \tilde{a}_1(t-t_0)^{-3} + \sum_{j=1}^{6} \widetilde{cz}_{j-3}(t-t_0)^{j-3} \quad \text{and} \quad y = -\frac{15}{8}(t-t_0)^{-2} + \sum_{j=1}^{6} \widetilde{cy}_{j-2}(t-t_0)^{j-2},$$

[4]In this case the original system is said to have the weak Painlevé property. For example, in the integrable case (iii) system (2) has the weak Painlevé property. The interesting example of a system with the weak Painlevé property is presented in [43].

where $\widetilde{cy}_j$ and $\widetilde{cz}_j$ are unknown constants, we obtain that the fourth system is:

$$\begin{cases} 557056\tilde{a}_1^6 + \tilde{a}_1^4\Big(15552000\lambda - 4860000\Big) + \\ \tilde{a}_1^2\Big(864000000\widetilde{cy}_2 + 108000000\lambda^2 - 67500000\lambda + 10546875\Big) = 0, \\ 818176\tilde{a}_1^4 + \tilde{a}_1^2\Big(15660000\lambda - 4893750\Big) - 810000000\widetilde{cy}_2 - 6328125 = 0. \end{cases} \tag{7}$$

As one can see this system does not include $\widetilde{cz}_1$, so $\widetilde{cz}_1$ is an arbitrary parameter (a constant of integration). At any value of λ this system can be solved as system in $\widetilde{cy}_2$ and $\tilde{a}_1$. We obtain new constant of integration $\widetilde{cz}_1$, but we must fix $\tilde{a}_1$, so number of constants of integration is equal to 2. It is easy to verify that $\widetilde{cy}_4$ is an arbitrary parameter, because the corresponding system is equivalent to one linear equation. So, using Painlevé test, we obtain an asymptotic solution which depends on three parameters: t_0, $\widetilde{cz}_1$ and $\widetilde{cy}_4$.

4 New asymptotic solutions

Now it is easy to obtain asymptotic solutions with arbitrary accuracy. For given values of λ one has to choose $\tilde{a}_1$ as one of the roots of system (7). After this the coefficients $\widetilde{cz}_j$ and $\widetilde{cy}_j$ as functions of $\widetilde{cz}_4$ and $\widetilde{cy}_6$ can be found automatically due to computer algebra system **REDUCE** [41, 42]. For some values of λ asymptotic solutions have been found as the following series (without loss the generality we can put $t_0 = 0$):

$$z = \tilde{a}_1 t^{-3} + \sum_{j=1}^{50} \widetilde{cz}_{j-3} t^{j-3} \qquad \text{and} \qquad y = -\frac{15}{8}t^{-2} + \sum_{j=1}^{50} \widetilde{cy}_{j-2} t^{j-2}.$$

For example, if $\lambda = \frac{1}{9}$, then system (7) has the following solutions ($\tilde{a}_1 \neq 0$):

$$\left\{\tilde{a}_1 = \pm\frac{25\sqrt{2}}{16}, \quad \widetilde{cy}_2 = -\frac{1819}{663552}\right\}, \qquad \left\{\tilde{a}_1 = \pm\frac{25i\sqrt{13}}{8\sqrt{374}}, \quad \widetilde{cy}_2 = -\frac{8700683}{1364926464}\right\}.$$

If $\tilde{a}_1 = \pm\frac{25\sqrt{2}}{16}$, then we obtain that at real values of parameters solutions are real:

$$\begin{aligned} z = &\tfrac{25\sqrt{2}}{16}t^{-3} + \tfrac{125}{192}t^{-2} + \tfrac{25\sqrt{2}}{768}t^{-1} + \tfrac{1625}{82944} + \widetilde{cz}_1 t + \\ &+\Big(\tfrac{21845}{47775744} - \tfrac{\sqrt{2}}{6}\widetilde{cz}_1\Big)t^2 + \Big(\tfrac{437425\sqrt{2}}{9172942848} - \tfrac{25\sqrt{2}}{48}\widetilde{cy}_4 - \tfrac{191}{3456}\widetilde{cz}_1\Big)t^3 + \ldots, \\ y = &-\tfrac{15}{8}t^{-2} + \tfrac{5\sqrt{2}}{32}t^{-1} - \tfrac{205}{2304} + \tfrac{115\sqrt{2}}{13824}t - \tfrac{1819}{663552}t^2 + \\ &+\Big(\tfrac{1673\sqrt{2}}{11943936} + \tfrac{1}{6}\widetilde{cz}_1\Big)t^3 + \widetilde{cy}_4 t^4 + \Big(\tfrac{1044461\sqrt{2}}{220150628352} - \tfrac{19}{9216}\widetilde{cz}_1 - \tfrac{1}{2}\widetilde{cy}_4\Big)t^5 + \ldots \end{aligned} \tag{8.1}$$

and

$$\begin{aligned} z = &-\tfrac{25\sqrt{2}}{16}t^{-3} + \tfrac{125}{192}t^{-2} - \tfrac{25\sqrt{2}}{768}t^{-1} + \tfrac{1625}{82944} + \widetilde{cz}_1 t + \\ &+\Big(\tfrac{21845}{47775744} + \tfrac{\sqrt{2}}{6}\widetilde{cz}_1\Big)t^2 + \Big(\tfrac{437425\sqrt{2}}{9172942848} - \tfrac{25\sqrt{2}}{48}\widetilde{cy}_4 - \tfrac{191}{3456}\widetilde{cz}_1\Big)t^3 + \ldots, \\ y = &-\tfrac{15}{8}t^{-2} - \tfrac{5\sqrt{2}}{32}t^{-1} - \tfrac{205}{2304} - \tfrac{115\sqrt{2}}{13824}t - \tfrac{1819}{663552}t^2 - \\ &-\Big(\tfrac{1673\sqrt{2}}{11943936} + \tfrac{1}{6}\widetilde{cz}_1\Big)t^3 + \widetilde{cy}_4 t^4 - \Big(\tfrac{1044461\sqrt{2}}{220150628352} - \tfrac{19}{9216}\widetilde{cz}_1 - \tfrac{23\sqrt{2}}{384}\widetilde{cy}_4\Big)t^5 + \ldots \end{aligned} \tag{8.2}$$

Numerical analysis shows [44] that coefficients tend to zero very rapidly when the absolute values of the parameters are less than unit.

5 The connection between asymptotic solutions and exact solutions

E.I. Timoshkova [20] found that solutions of the following equation

$$y_t^2 = \tilde{\mathcal{A}} y^3 + \tilde{\mathcal{G}} y^{5/2} + \tilde{\mathcal{B}} y^2 + \tilde{\mathcal{E}} y^{3/2} + \tilde{\mathcal{C}} y + \tilde{\mathcal{D}}, \qquad (9)$$

where $\tilde{\mathcal{A}} = \frac{-32}{15}$, $\tilde{\mathcal{D}} = 0$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{E}}$ are some depending on λ constants, satisfy (3) at $C = -\frac{16}{5}$. Changing variables: $y = \varrho^2$, we obtain the following equation:

$$\varrho_t^2 = \frac{1}{4}\Big(\tilde{\mathcal{A}}\varrho^4 + \tilde{\mathcal{G}}\varrho^3 + \tilde{\mathcal{B}}\varrho^2 + \tilde{\mathcal{E}}\varrho + \tilde{\mathcal{C}}\Big). \qquad (10)$$

Equation (10) is of P-type. The general solution of equation (10) has only one arbitrary parameter t_0 and can be expressed in elliptic functions [45].

For all values of λ the trajectories of motion are given by the following equation:

$$x^2 \equiv z = -\frac{1}{2}\left(\frac{5}{2}\tilde{\mathcal{G}} y^{3/2} + 2(\tilde{\mathcal{B}}+1)y + \frac{3}{2}\tilde{\mathcal{E}} y^{1/2} + \tilde{\mathcal{C}}\right). \qquad (11)$$

Let us compare the asymptotic solutions (8.1) and (8.2) with these exact solutions. It is easy to verify, for example, with the help of the Painlevé test of equation (10), that at points of singularities these solutions and our asymptotic solutions have the same asymptotic behavior. It means that for some values of parameters $\widetilde{cz}_1$ and $\widetilde{cy}_4$ our Laurent series give exact solutions and, hence, converge.

Let us consider in detail the case of $\lambda = \frac{1}{9}$. In this case the following solutions have been found [20]:

$$x^2 \equiv z = -\frac{y}{3}\left\{\frac{5}{3} - 2i\sqrt{\frac{5y}{3}}\right\}, \qquad (12)$$

where y satisfies the following equation:

$$y_t^2 + \frac{32}{15}y^3 + \frac{4}{9}y^2 \pm \frac{8i}{\sqrt{135}}y^{5/2} = 0. \qquad (13)$$

For sign "+":
$$y = -\frac{5}{3\left(1 - 3\sin\left(\frac{t-t_0}{3}\right)\right)^2} \quad \text{and} \quad z = \frac{25\left(1 - \sin\left(\frac{t-t_0}{3}\right)\right)}{9\left(1 - 3\sin\left(\frac{t-t_0}{3}\right)\right)^3}. \qquad (14.1)$$

For sign "−":
$$y = -\frac{5}{3\left(1 + 3\sin\left(\frac{t-t_0}{3}\right)\right)^2} \quad \text{and} \quad z = \frac{25\left(1 + \sin\left(\frac{t-t_0}{3}\right)\right)}{9\left(1 + 3\sin\left(\frac{t-t_0}{3}\right)\right)^3}. \qquad (14.2)$$

It is easy to verify that the series (8.1) with $\widetilde{cy}_4 = -\frac{858455}{12039487488}$ and $\widetilde{cz}_1 = \frac{3205\sqrt{2}}{3981312}$ are the Laurent series of (14.1) and the series (8.2) with $\widetilde{cy}_4 = -\frac{858455}{12039487488}$ and $\widetilde{cz}_1 = -\frac{3205\sqrt{2}}{3981312}$ are the Laurent series of (14.2). So, our asymptotic series converge for these values of parameters. It is possible that our asymptotic series with other values of parameters converge as well. I plan to analyze this question in future.

CONCLUSIONS

Using the Painlevé analysis one can not only define whether the system is integrable or not, but find special asymptotic solutions even in nonintegrable case as well. We have found special solutions of the Hénon–Heiles system with $C=-\frac{16}{5}$. These special solutions are asymptotic solutions and depend on three parameters. The obtained solutions with some values of two parameters coincide with the one-parameter set of exact solutions.

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