Exact solution of the one-atom maser model

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Abstract

An exact analytical expression for a density matrix of atom + field system is obtained taking into account relaxation of photons and frequency detuning (Jaynes-Cummings model with dissipation). The photon mean number, atom population inversion and Fano Q - factor depending on initial states of the field and various parameters of the one-atom maser are investigated.

1 Introduction

The Jaynes-Cummings model (JCM) [1] of a two-level atom interacting with the photon mode in the ideal cavity plays an essential role in modern quantum optics. It predicts many essential nonclassical effects such as Rabi oscillations in presence of vacuum cavity field, revivals and collapses of Rabi oscillations in cavity coherent field [2, 3] etc. All these predictions were tested successfully in experiments, see, for example, review articles [4, 5]. In reality, one does experiments with rubidium Rydberg atoms pumped by laser excitation at the upper level of the maser transition in cavities with very high but finite quality Q. The atom-field interaction is probed by observing the populations at the upper and lower levels after the atoms have left the cavity. The electromagnetic field in cavity consists of few photons depending on the flux of atoms. The maser dynamics treated with the JCM was experimentally investigated by selecting and varying the velocity of the atoms. The counting statistics of the cavity emerging atoms allowed one to observe the nonclassical behavior of the cavity field. In important paper the effect of cavity leakage on emission and absorption of two-level atom was studied exactly by solving the density matrix equation with the Laplace transform. When an atom in upper state passes through the cavity with slow velocity and the cavity mode is supposed to be prepared in the vacuum state, only four states from infinite set are important for the maser dynamics, and only photon mode relaxation was taken into account.

2 Jaynes-Cummings model with dissipation

We study a stationary two-level atom in non-ideal cavity. Let us consider the Hamiltonian

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_B + \hat{H}_{AF} + \hat{H}_{BF}. \tag{1}$$

Here $\hat{H}_A = \hbar\omega_0 \hat{S}_3$ - Hamiltonian of a free atom,

$\hat{H}_F = \hbar\omega(\hat{a}^+\hat{a} + 1/2)-$ photon mode Hamiltonian,

$\hat{H}_B = \sum\limits_{\alpha=1}^{?} \hbar\tilde{\omega}_\alpha(\hat{c}_\alpha^+\hat{c}_\alpha + 1/2)-$ "bath" for photons Hamiltonian,

$\hat{H}_{AF} = \hbar g(\hat{a}^+\hat{S}^- + \hat{a}\hat{S}^+)-$ Atom – field interaction operator,

$\hat{H}_{BF} = \sum\limits_{\alpha=1}^{?} (\hat{c}_\alpha^+\hat{a}f_\alpha + \hat{c}_\alpha\hat{a}^+\bar{f}_\alpha)-$ "Bath" - photons interaction Hamiltonian. So equation for density matrix is as follows:

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = \left[\hat{H}, \hat{\rho}\right]. \tag{2}$$

We will find out the density matrix in the form: $\hat{\rho}(t) = \hat{\rho}_{AF}(t) \otimes \hat{\rho}_B(0)$.

Using this irreversibility hypothesis allows us to write the master equation for the density matrix:

$$\frac{\partial\hat{\rho}_{AF}}{\partial t} = \frac{1}{i\hbar}\left[\hat{H}_{AF}^{(I)}, \hat{\rho}_{AF}\right] + \hat{L}_F\hat{\rho}_{AF}, \tag{3}$$

where

$$\begin{aligned}\hat{L}_F\hat{\rho}_{AF} \;\;=\;\; & -\frac{\gamma}{2}\Big((\langle n\rangle + 1)[\hat{a}^+\hat{a}\hat{\rho}_{AF} - 2\hat{a}\hat{\rho}_{AF}\hat{a}^+ + \hat{\rho}_{AF}\hat{a}^+\hat{a}] \\ & +\langle n\rangle[\hat{a}\hat{a}^+\hat{\rho}_{AF} - 2\hat{a}^+\hat{\rho}_{AF}\hat{a} + \hat{\rho}_{AF}\hat{a}\hat{a}^+]\Big)\end{aligned} \tag{4}$$

In the last expression γ is a damping constant, and parameter $\langle n\rangle$ is as usual a mean number of quanta in heat bath.

$\hat{\rho}_{AF} = \sum\limits_{\mu,\nu=1}^{2} \int \frac{d^2\alpha}{\pi} P_{\mu\nu}(\alpha, t)|\alpha\rangle\langle\alpha| \otimes |\mu\rangle\langle\nu|.$

A solution for reduced density matrix will be considered in ***P***-representation of Glauber – Sudarshan. For this purpose, let us introduce the density matrix in the form:

$$\hat{\rho}_{AF} = \int \sum_{\mu,\eta} P_{\mu\eta}\left(\bar{\alpha}, \alpha, t\right)|\alpha\rangle\langle\alpha| \otimes |\mu\rangle\langle\eta|\frac{d^2\alpha}{\pi}, \tag{5}$$

where $|\alpha\rangle$ Glauber coherent state and $|\mu\rangle(|\nu\rangle)-$ eigen-states of two-level atom (μ,ν =1,2).

In these variables we get the system of Fokker-Planck-like equations for the components of P-symbol.

For further calculations it is convenient to construct from the elements of the P-symbol a vector with four components. Then, the Fokker-Planck equation will be rewritten as:

$$\frac{\partial}{\partial t}\vec{\mathrm{P}} = \hat{L}_0\vec{\mathrm{P}} + \hat{L}_1\vec{\mathrm{P}}, \tag{6}$$

where $\vec{\mathrm{P}}$ - a column defined as

$$\vec{\mathrm{P}} = \begin{pmatrix} P_{11} & P_{12} & P_{21} & P_{22} \end{pmatrix},$$

operators $\hat{L}_0, \hat{L}_1$ - from Eq.(6) are

$$\hat{L}_0 = -\frac{\gamma_F}{2}\left\{\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\bar{\alpha}}\bar{\alpha} + 2\langle n_F\rangle\frac{\partial^2}{\partial\bar{\alpha}\partial\alpha}\right\}\hat{1},$$

$$\hat{L}_1 = -ig\begin{pmatrix} 0 & -\alpha + \frac{\partial}{\partial\bar{\alpha}} & \bar{\alpha} - \frac{\partial}{\partial\alpha} & 0 \\ -\bar{\alpha} & 0 & 0 & \bar{\alpha} - \frac{\partial}{\partial\alpha} \\ \alpha & 0 & 0 & -\alpha + \frac{\partial}{\partial\bar{\alpha}} \\ 0 & \alpha & -\bar{\alpha} & 0 \end{pmatrix}.$$

Here $\hat{I}$ is the unity 4×4 matrix in space of atomic variables.

Going to the representation:

$$\hat{\rho}' = \exp\left\{i\hat{H}_{AF}^{(I)}t/\hbar\right\}\hat{\rho}_{AF}\exp\left\{-i\hat{H}_{AF}^{(I)}t/\hbar\right\}, \tag{7}$$

one can easily get

$$\frac{\partial\hat{\rho}'}{\partial t} = -\frac{\gamma}{2}\Big\{(\langle n\rangle+1)\left[\hat{a}'^{+}\hat{a}'\hat{\rho}' - 2\hat{a}'\hat{\rho}'\hat{a}'^{+} + \hat{\rho}'\hat{a}'^{+}\hat{a}'\right] + \langle n\rangle\left[\hat{a}'\hat{a}'^{+}\hat{\rho}' - 2\hat{a}'^{+}\hat{\rho}'\hat{a}' + \hat{\rho}'\hat{a}'\hat{a}'^{+}\right]\Big\} \tag{8}$$

here $\hat{a}' = \exp\left\{i\hat{H}_{AF}^{(I)}t/\hbar\right\}\hat{a}\exp\left\{-i\hat{H}_{AF}^{(I)}t/\hbar\right\}$, $\hat{a}'^{+} = \exp\left\{i\hat{H}_{AF}^{(I)}t/\hbar\right\}\hat{a}^{+}\exp\left\{-i\hat{H}_{AF}^{(I)}t/\hbar\right\}$, $|\alpha'\rangle = \exp\left\{i\hat{H}_{AF}^{(I)}t/\hbar\right\}|\alpha\rangle$

It is important to note that the pattern of eigen-states for $|\alpha\rangle$ and $|\alpha'\rangle$ satisfy the same relation as vector $|\alpha$¿:

$$\hat{a}'|\alpha'\rangle = \alpha|\alpha'\rangle. \tag{9}$$

Obviously the master equation in this representation is identical to the master equation for photon mode in cavity. So we can easily write its solution.

From the master equation we can write the equation in ***P***-representation, which is the Fokker-Plank equation for photon mode in cavity

$$\frac{\partial P}{\partial t} = \frac{\gamma}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\bar{\alpha}}\bar{\alpha} + 2\langle n\rangle\frac{\partial^2}{\partial\alpha\partial\bar{\alpha}}\right)P \tag{10}$$

and well known solution of this equation

$$P(\alpha,t) = \int\frac{d^2\alpha_0}{\pi}K(\alpha,t|\alpha_0,0)\,P(\alpha_0,t), \tag{11}$$

where

$$K(\alpha,t|\alpha_0,0) = \frac{1}{\langle n\rangle\left(1-e^{-\gamma t}\right)}exp\left\{-\frac{\left|\alpha-\alpha_0e^{-\gamma t/2}\right|^2}{\langle n\rangle\left(1-e^{-\gamma t}\right)}\right\}. \tag{12}$$

The function $P(\alpha_0,t)$ presents the ***P***-symbol of initial state of field mode.

To get an appropriate form of the solution we should come back to $\hat{\rho}$. To this goal the expression

$$|\alpha'\rangle = \exp\left\{i\hat{H}^{(I)}_{AF}t/\hbar\right\}|\alpha\rangle$$

shouldbedetailed.

We can expand $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$ in Fock basis; also we will consider atoms in upper initial state

$$\hat{\rho}_0 = \begin{pmatrix} \hat{\rho}_F(0) & 0 \\ 0 & 0 \end{pmatrix}. \tag{13}$$

We can write a solution for the density matrix (exp $\left\{i\hat{H}^{(I)}_{AF}t/\hbar\right\}|n\rangle$ a well-known solution of JC-model in ideal cavity

$$\begin{aligned}\hat{\rho}_{AF} \;=\; & \sum_{n,m=0}^{\infty}\frac{F_{nm}(\alpha_0,t)}{\sqrt{n!m!}}(\cos(\Omega_n t/2)+i\cos\theta_n\sin(\Omega_n t/2))(\cos(\Omega_m t/2) \\ & -i\cos\theta_m\sin(\Omega_m t/2))|n,1\rangle\langle m,1|+\end{aligned}$$

$$\sum_{n,m=0}^{\infty}\frac{F_{nm}(\alpha_0,t)}{\sqrt{n!m!}}\left(Cos(\Omega_n t/2)+iCos\theta_n Sin(\Omega_n t/2)\right)\left(iSin\theta_m Sin(\Omega_m t/2)\right)|n,1\rangle\langle m+1,2|+$$

$$\sum_{n,m=0}^{\infty}\frac{F_{nm}(\alpha_0,t)}{\sqrt{n!m!}}\left(-iSin\theta_n Sin(\Omega_n t/2)\right)\left(Cos(\Omega_m t/2)-iCos\theta_m Sin(\Omega_m t/2)\right)|n+1,2\rangle\langle m,1|+$$

$$\sum_{n,m=0}^{\infty}\frac{F_{nm}(\alpha_0,t)}{\sqrt{n!m!}}\left(Sin\theta_n Sin(\Omega_n t/2)\right)\left(Sin\theta_m Sin(\Omega_m t/2)\right)|n+1,2\rangle\langle m+1,2|. \tag{14}$$

Here for $m \geq nF_{nm}(\alpha_0,t) = \frac{s^n}{(s+1)^{m+1}}e^{-\frac{|\alpha_0|^2e^{-\gamma t}}{s+1}}n!\bar{\alpha}_0^{m-n}e^{-\gamma t/2(m-n)}L_n^{m-n}\left(-\frac{|\alpha_0|^2e^{-\gamma t}}{s(s+1)}\right)$,
and for $m < nF_{nm}(\alpha_0,t) = \frac{s^m}{(s+1)^{n+1}}e^{-\frac{|\alpha_0|^2e^{-\gamma t}}{s+1}}m!\alpha_0^{m-n}e^{-\gamma t/2(n-m)}L_m^{n-m}\left(-\frac{|\alpha_0|^2e^{-\gamma t}}{s(s+1)}\right)$.
$L_m^{n-m}(x)-$ a generalized Laguerre polynomial, and parameter $s = \langle n\rangle\left(1-e^{-\gamma t}\right)$.

3 Results of computer simulations and discussion

The derived formula for the P-symbol of the density matrix is exact, but it contains some summations. It is easy to show that all this series (with definite coefficients) are convergent and lead to computable expressions.

We perform the computer simulations of time dependence for some interesting for micromaser problem parameters: the mean number of photons in cavity, the Fano Q-factor, and inversion of the atomic population. The computations were made with different initial field state (n-photon Fock state, coherent state and chaotic (thermal) photon state. We present these computations in Figs. 1,2,3,4. Note that for the convenience of plotting some curves are shown with displacements.

4 Summary

The obtained results for observables are in good agreement with recent numerical and experimental data of Walther and Haroche groups, see, for example the review article [5]. Within the framework of the developed approach it is planned:

to take into account movement of atom (atoms) in cavity.

to generalize model by taking into account non-resonant terms in atom-field interaction Hamiltonian.

Detailed derivation of density matrix expression and generalizations will be published elsewhere.

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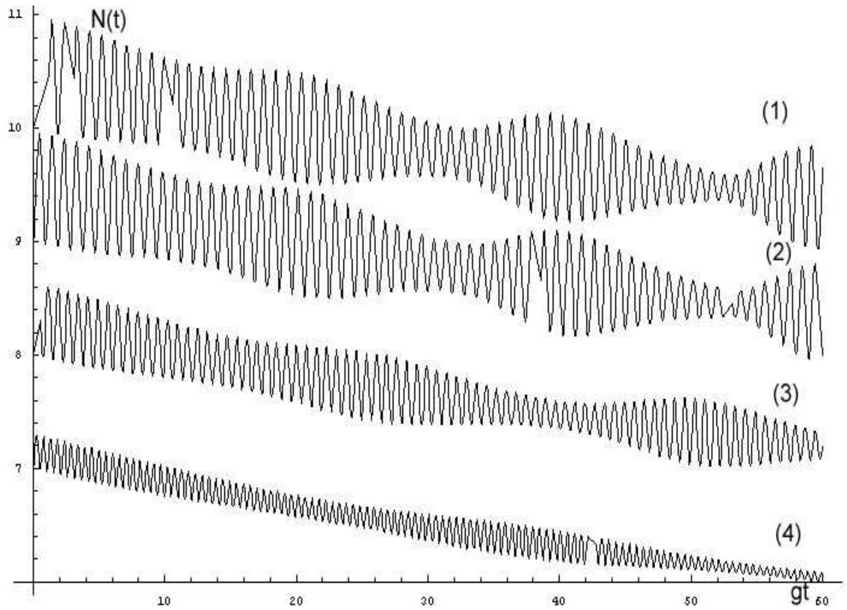

Figure 1: Time dependence of the mean photon number when photon mode is prepared in state with definite number of quanta n= 10; $\langle n \rangle$ =0.15, g=1. Curves **(1)** – **(5)** represent different detunings $\Delta\omega$. **(1):** $\Delta\omega$ = 0,00001; **(2):** $\Delta\omega$ =1; **(3):** $\Delta\omega$ = 5; $\Delta\omega$. **(5):** $\Delta\omega$ = 10.

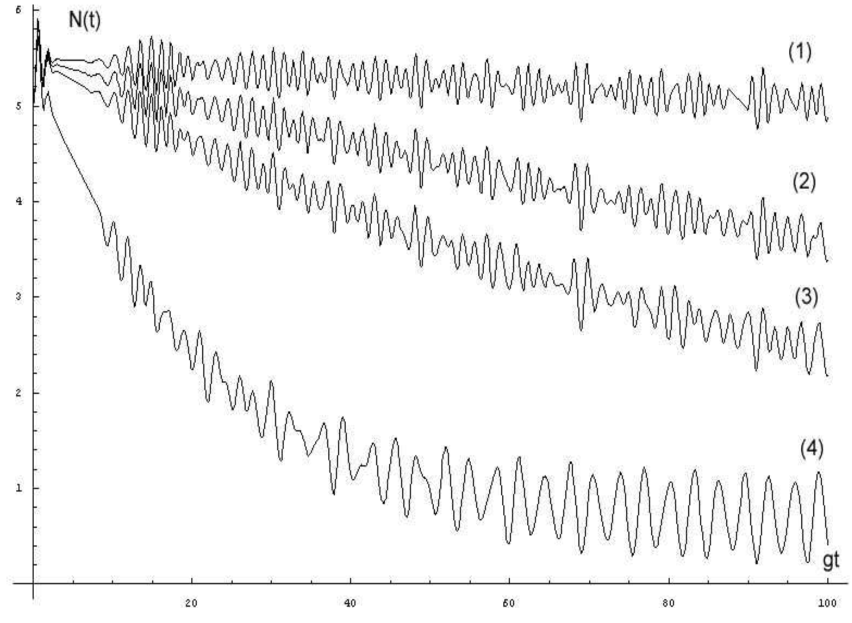

Figure 2: Time dependence of the mean photon number when photon mode is prepared in coherent state with $|\alpha|^2$ = 5, $\langle n \rangle$ =0.15, g=1. Curves **(1)** – **(5)** represent different damping constants γ. **(1)** γ = 0,001; **(2)** γ=0.005; **(3)** γ = 0.01;**(5)** γ = 0.05.

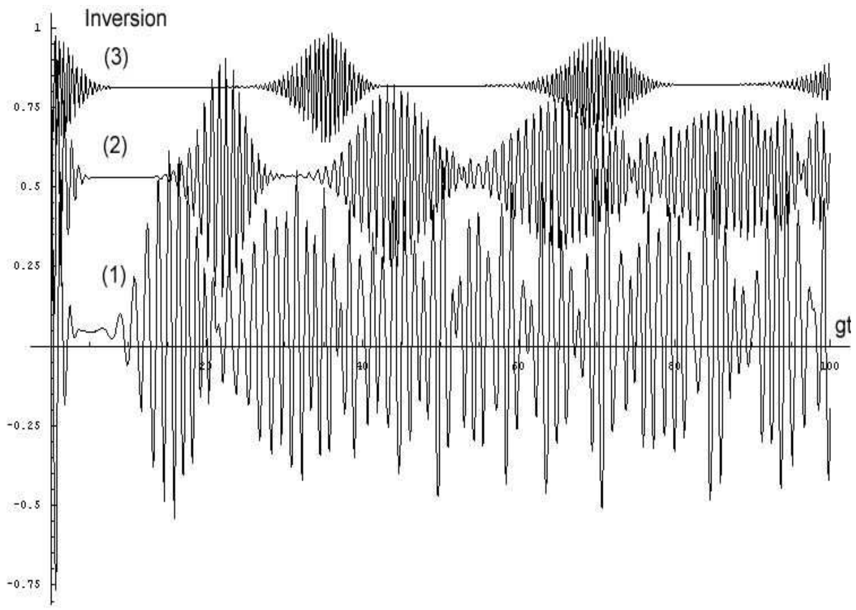

Figure 3: Time dependence of the atomic population inversion. The photon mode is initially prepared in coherent state with $|\alpha|^2 = 5$; $\langle n \rangle$ =0.15, g=1, γ=0.001. Curves **(1)** – **(3)** represent different detunings $\Delta\omega$. **(1):** $\Delta\omega = 1$; **(2):** $\Delta\omega$ =5; **(3):** $\Delta\omega = 10$.

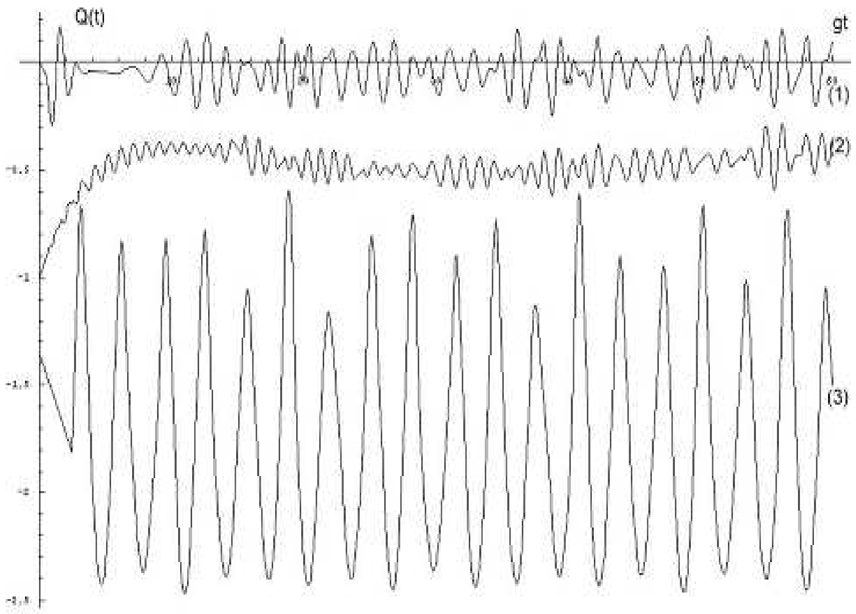

Figure 4: Time dependence of the Fano Q-factor for different initial field, $\langle n \rangle$ =0.15, g=1, γ=0.001. **(1):** Photon mode is prepared in coherent state with $|\alpha|^2 = 5$; **(2):** Photon mode is prepared in initial Fock state with n = 10; **(3):** Photon mode is prepared in chaotic state with ¡n¿ = 0.1. Here the curve **(3)** shows the function Q- 3/2