Linearized gravity and quantum gravity effects in the RS2 model

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Abstract

In the following we investigate gravitational field and effects of quantum gravity outside topological defects with angular deficit in the Randall-Sundrum infinite brane world.

1 Introduction

All present-day approaches to the fundamental theory are based on the idea that our Universe is a subspace of some higher dimensional space-time. The original theories of the Kaluza-Klein type space-time were assumed to be the direct product of a four-dimensional space-time and a homogeneous compact manifold of internal coordinates. To obtain an appropriate four-dimensional low energy effective theory, it was supposed that the size of extra dimensions is of the Plank scale. So, for energies accessible in an earth laboratory the modes of matter fields with a momentum in the direction of extra dimensions can not be exited thus shielding the existence of internal space.

At present time the emphasis in the development of multidimensional theories is shifted toward the brane world picture [1]. In this scenario, to preserve the consistency with high-energy physics experiments, it is supposed that all Standard Model fields are confined to the four-dimensional submanifold (brane) of a fundamental higher-dimensional bulk. One of the most attractive features of the brane world picture is that this scenario enables gravity to become strong at energies of the TeV scale. So, the effects of embedding can, at least in principle, be observed in the coming experiments.

In our paper we investigate the linearized gravitational field of topological defects localized on a brane and bulk induced quantum gravity effects near defects.

We use the metric with the signature $(- + \cdots +)$ and the system of units $\hbar = c = 1$.

2 Linearized gravity of brane defects

We will start by reviewing the formulation of linearized RS2 gravity [2] when matter is present [3] and in the case of $(n+1)$ space-time dimensions (the latter is necessary because of the use of the dimensional regularization method in the next section).

The metric can be written as

$$ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu + 2N_\mu dx^\mu dy + (1+\phi) dy^2 \ , \tag{1}$$

where

$$\hat{g}_{\mu\nu} = e^{-2k|y|}(\eta_{\mu\nu} + \gamma_{\mu\nu}) \ , \tag{2}$$

x^μ, $\mu = 0,1,...,n-1$, are the coordinates on the brane and additional coordinate y is chosen so that the brane is kept straight and localized at $y=0$. $\hat{g}_{\mu\nu}$ gives the induced metric on the hypersurfaces $y = \text{const}$. The length k^{-1} is related to the $(n+1)$-dimensional bulk cosmological constant by $\Lambda = -n(n-1)k^2/2$ and gives the characteristic length scale of the bulk corrections to the usual n-dimensional Einstein's gravity.

In the linear approximation $\gamma_{\mu\nu}$, N_μ and ϕ are taken to be small, and gauge conditions for the brane to remain at $y=0$ read

$$N_\mu = -\frac{\operatorname{sgn} y}{2nk}\gamma_{,\mu} \ , \quad \phi = -\frac{\operatorname{sgn} y}{nk}\gamma_{,y} \ , \quad \tilde{\gamma}_{\mu\nu}{}^{,\mu} = 0 \ , \tag{3}$$

where $\gamma = \eta^{\mu\nu}\gamma_{\mu\nu}$ and $\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - (1/n)\eta_{\mu\nu}\gamma$ is the traceless part of $\gamma_{\mu\nu}$.

Proceeding along the same line as in [3] for $k>0$ and $q^2>0$ we obtain

$$\tilde{\gamma}_{\mu\nu}(x,0) = 8\pi G_{bulk}\int \frac{d^nq}{(2\pi)^n}e^{iqx}\frac{K_{n/2}\left(|q|/k\right)}{K_{n/2-1}(|q|/k)}\frac{W_{\mu\nu}(q)}{|q|} \tag{4}$$

with

$$W_{\mu\nu}(q) = t_{\mu\nu}(q) - \frac{1}{n-1}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)t(q) \tag{5}$$

and

$$\gamma(x,0) = 8\pi G_{bulk}\frac{nk}{n-1}\int \frac{d^nq}{(2\pi)^n}e^{iqx}\frac{t(q)}{q^2} \ , \tag{6}$$

where $t_{\mu\nu}$ is the energy-momentum tensor of a matter $t = \eta^{\mu\nu}t_{\mu\nu}$. G_{bulk} is the $(n+1)$-dimensional fundamental Newton's constant, which determines the effective n-dimensional brane gravitational constant by the relation $G_{brane} = (n-2)kG_{bulk}/2$ [4].

Now we are in a position to consider the particular case of a global monopole in the physical case $n=4$. Corresponding energy-momentum tensor has the form

$$t_{00} = \frac{\eta^2}{R^2} \ , \quad t_{ik} = -\eta^2\frac{x_ix_k}{R^4} \ , \quad R^2 = x^ix^i \ , \quad i,k = 1,2,3 \ , \tag{7}$$

where η is the energy scale of the corresponding phase transition.

From the Eqs (4), (5) and (6) we get that on the brane

$$\begin{gathered}\tilde{\gamma}_{00}(x) = \frac{16G_4\eta^2\pi^3}{3k}\int \frac{d^3q}{(2\pi)^3}\frac{e^{iqx}}{q^2}\frac{K_2\left(q/k\right)}{K_1(q/k)} \ , \\ \tilde{\gamma}_{ij}(x) = \frac{8G_4\eta^2\pi^3}{3k}\int \frac{d^3q}{(2\pi)^3}\frac{e^{iqx}}{q^2}\left(\delta_{ij} - \frac{q_iq_j}{q^2}\right)\frac{K_2\left(q/k\right)}{K_1(q/k)} \ , \\ \gamma(x) = \frac{8G_4\eta^2\pi^3}{3k}\int \frac{d^3q}{(2\pi)^3}\frac{e^{iqx}}{q^3} \ .\end{gathered} \tag{8}$$

Changing to the spherical coordinates, and evaluating the angular integrals simplifies the expressions for the metric perturbation to

$$\gamma_{00}(R) = \frac{16G_4\eta^2\pi^3}{3k}\left(I_1 - 2kI_0\right) \ , \quad \gamma_{RR}(R) = \frac{8G_4\eta^2\pi^3}{3k}(I_1 + 4kI_0 + \partial_R\partial_RI_2) \ ,$$

$$\gamma_{\theta\theta}(R) = \frac{8G_4\eta^2\pi^3}{3k}R^2(I_1 + 4kI_0 + \frac{1}{R}\partial_R I_2) \ , \quad \gamma_{\phi\phi}(R,\theta) = \sin^2\theta\gamma_{\theta\theta} \ , \tag{9}$$

with

$$I_0(R) = \frac{2}{(2\pi)^2}\int\limits_0^\infty \frac{dq}{q}\frac{\sin(qR)}{qR} \ , \quad I_1(R) = \frac{2}{(2\pi)^2}\int\limits_0^\infty dq\frac{\sin(qR)}{qR}\frac{K_2\,(q/k)}{K_1(q/k)} \ ,$$

$$I_2(R) = \frac{2}{(2\pi)^2}\int\limits_0^\infty \frac{dq}{q^2}\frac{\sin(qR)}{qR}\frac{K_2\,(q/k)}{K_1(q/k)} \ . \tag{10}$$

Let us introduce a new radial coordinate r satisfying

$$\left[1 + \frac{8G_4\eta^2\pi^3}{3k}\left(I_1 + 4kI_0 + \frac{\partial_R I_2}{R}\right)\right]R^2 = \left[1 - 8\pi G_4\eta^2\,(1 + f_1(kr))\right]r^2 \ . \tag{11}$$

In therms of this new coordinate the line element of a monopole embedded in a 3-brane reads

$$d\hat{s}^2 = -dt^2[1 - 8\pi G_4\eta^2 f_2(kr)] + dr^2 + [1 - 8\pi G_4\eta^2(1 + f_1(kr))]r^2 d\Omega_2 \ , \tag{12}$$

where functions f_1 and f_2 are determined by the equations

$$\partial_x[xf_1(x)] = \frac{1}{6}\int\limits_0^\infty dq\frac{(\sin(qx) - qx\cos(qx))}{qx}\frac{K_0(q)}{K_1(q)} \ , \quad xf_1(x)|_{x\to\infty} = 0 \tag{13}$$

and

$$f_2(x) = \frac{1}{3}\int\limits_0^\infty dq\frac{\sin(qx)}{qx}\frac{K_0(q)}{K_1(q)} \ . \tag{14}$$

It can be verified, that when $k \to \infty$ both functions tend to zero, and the line element on the brane coincides with the one obtained in the membrane cosmology [5].

From Eqs. (12), (13) and (14) we see, that possibility of gravity to propagate in extra dimension changes the local geometry of the monopole space-time. One can show, that at large distances, as compared to $1/k$

$$\begin{aligned} d\hat{s}^2 = &-dt^2\left[1 - \frac{8\pi G_4\eta^2}{3k^2r^2}\log(2kr)\right] + dr^2 + \\ &+ \left[1 - 8\pi G_4\eta^2\left(1 - \frac{1}{6k^2r^2}\,(2\log(2kr) + 1)\right)\right]r^2 d\Omega_2 \ , \end{aligned} \tag{15}$$

while if $r \ll 1/k$

$$d\hat{s}^2 = -dt^2\left[1 - \frac{4\pi^2 G_4\eta^2}{3kr}\right] + dr^2 + \left[1 - 8\pi G_4\eta^2\left(1 + \frac{\pi\log(kr)}{12kr}\right)\right]r^2 d\Omega_2 \ . \tag{16}$$

Thus, in contrast to the case of four space-time dimensions, gravitational potential of a brane monopole is not equal to zero. It can be shown, that corresponding linearized intrinsic Ricci tensor is the sum of contribution from the higher dimensional bulk and contribution which for $n = 4$ coincides with the result obtained in the $4D$ Einstein's theory. At the same time in the lowest order the scalar curvature

$$\hat{R}^{mon} = 16\pi G_4 \frac{\eta^2}{r^2} \tag{17}$$

and contains no contribution from the extra dimension.

In the case of a straight cosmic string

$$T_t^t = T_z^z = \mu \frac{\delta(\rho - \epsilon)}{2\pi\rho} \ , \quad \epsilon \to +0 \ , \tag{18}$$

ρ is the cylindrical radial coordinate.

Proceeding along the same lines as above we get that at large distances (see also [6])

$$ds^2 = (-dt^2 + dz^2)\left[1 - \frac{4G_4\mu}{3k^2\rho^2}\right] + d\rho^2 + \left[1 - 8G_4\mu + \frac{16G_4\mu}{3k^2\rho^2}\right]\rho^2 d\theta^2 \ , \tag{19}$$

where $\mu \sim \eta^2$ is the energy per unit length of the string. Thus corresponding space is approximately conical, but the brane corrections give the string a small attractive potential.

Near the brane string the metric reads

$$ds^2 = (-dt^2 + dz^2)\left[1 - \frac{4G_4\mu}{3k\rho}\right] + d\rho^2 + \left[1 - 8G_4\mu - \frac{8G_4\mu}{3k^2\rho^2}\log(k\rho)\right]\rho^2 d\phi^2 \ . \tag{20}$$

If $k^{-1} \sim 1$ mm and if the energy scale of the cosmological phase transition is about $\sim 10^{16}$ GeV, the dimensionless quantities $G_4\eta^2/k^2r^2$ and $G_4\mu/k^2\rho^2$ are too small at astronomical distances for the solutions (15) and (19) to be of interest for cosmology. On the other hand, differences between local and global structures of the brane and membrane monopoles and strings may be of importance for the processes near the core, such as the effect of vacuum polarization.

3 Vacuum fluctuations in the brane background

To calculate the vacuum expectation value $\langle T_\mu^\nu \rangle_{vac}^{ren}$, let us start from the formal expression

$$\langle T_\mu^\nu \rangle_{vac} = \lim_{x' \to x} D_\mu^{\nu'}(x, x') G^E(x, x') \ , \tag{21}$$

where $D_\mu^{\nu'}$ is a differential operator whose explicit form is determined by the classical expression for the stress-energy tensor and G^E is the Euclidean Green's function.

For the particular case of a massless minimally coupled scalar field and in the lowest order with respect to the gravitational constant, we have

$$\hat{D}_\mu^{\nu'} = \hat{\nabla}_\mu^{\nu'} - \frac{1}{2}\delta_\mu^\nu \hat{\nabla}_\lambda^{\lambda'} \ . \tag{22}$$

In our earlier paper [4] we have shown that the first-order correction to the flat space Euclidean Green's function on the brane $G_E^{(1)} = G_{Eins} + G_{bulk}$, where

$$G_{Eins}(x,x') = 8\pi G_{balk}\frac{(n-2)}{2}k\int\frac{d^nq}{(2\pi)^n}\frac{e^{iqx}}{q^2}\int\frac{d^np}{(2\pi)^n}\frac{e^{-ip(x-x')}}{p^2(p-q)^2}\times$$
$$\times\left[2W_{\mu\nu}(q)p^\mu p^\nu - \frac{p_\mu q^\mu}{n-1}t\right] , \tag{23}$$

coincides with the expression obtained for the first-order corrections to the Green's function in the n-dimensional Einstein's gravity [7], where

$$G_{bulk}(x,x') = 8\pi G_{n+1}\int\frac{d^nq}{(2\pi)^n}e^{iqx}\frac{K_{n/2-2}\left(|q|/k\right)}{|q|K_{n/2-1}\left(|q|/k\right)}\int\frac{d^np}{(2\pi)^n}\frac{e^{-ip(x-x')}}{p^2(p-q)^2}W_{\mu\nu}(q)p^\mu p^\nu \tag{24}$$

is the contribution from the extra dimension.

Substituting expressions (23) and (24) into (21), we obtain, that vacuum averaged energy-momentum tensor consists of two parts $\langle T_\mu^\nu\rangle_{vac} = \langle T_\mu^\nu\rangle_{Einst} + \langle T_\mu^\nu\rangle_{bulk}$.

Subsequent calculations can be performed with the use of the dimensional regularization method along the same lines, as in the case of cosmic strings in the four-dimensional Einstein's theory [7] (see also [4]). We find that

$$\langle T_\mu^\nu\rangle_{Einst}^{reg} = 4\pi G_{balk}k\frac{\Gamma(2-D/2)\Gamma^2(D/2)}{(D+1)\Gamma(D)}\frac{\Lambda^{n-D}}{(4\pi)^{D/2}}\int\frac{d^nq}{(2\pi)^n}e^{iqx}|q|^{D-2}\times$$
$$\times\left[W_\mu^\nu(q) - \frac{(D-2)(D+1)}{2(D-1)}\left(\delta_\mu^\nu - \frac{q_\mu q^\nu}{q^2}\right)t(q)\right] , \tag{25}$$

and

$$\langle T_\mu^\nu\rangle_{bulk}^{reg} = 8\pi G_{bulk}\frac{\Gamma(2-D/2)\Gamma^2(D/2)}{4(D+1)(D/2-1)\Gamma(D)}\times$$
$$\times\frac{1}{(4\pi)^{D/2}}\int\frac{d^nq}{(2\pi)^n}e^{iqx}\frac{K_{D/2-2}\left(|q|/k\right)}{K_{D/2-1}(|q|/k)}|q|^n\left(\frac{|q|}{\Lambda}\right)^{D-n}W_\mu^\nu(q) , \tag{26}$$

with Λ being an arbitrary parameter with the dimension of mass, which must be introduced to restore the dimension of $\langle T_\mu^\nu\rangle_{vac}^{reg}$.

Vacuum averaged energy-momentum tensor is divergent when $D\to 4$. As it was shown in Ref. [8], in four space-time dimensions the divergent part of $\langle T_\mu^\nu\rangle$ for the massless scalar field reads

$$\langle T_\mu^\nu\rangle_{div} = -\frac{1}{(4\pi)^{D/2}}\left[\frac{1}{D-4} + \frac{1}{2}\left(\gamma + \log\left(\frac{\Lambda_0^2}{\Lambda'^2}\right)\right)\right]\left(\frac{1}{90}H_\mu^\nu - \frac{1}{90}{}^{(2)}H_\mu^\nu + \frac{1}{36}{}^{(1)}H_\mu^\nu\right) . \tag{27}$$

In the last expression Λ_0 is the infrared cutoff which must be introduced in the case of massless fields and Λ' is an arbitrary constant. Explicit expressions for $H_{\mu\nu}$, ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ are more or less complicated (see for example Ref. [9]), but in the linear approximation (27) can be reduced to the following form

$$\langle T_\mu^\nu\rangle_{div} = -\frac{1}{90(4\pi)^{D/2}}\left[\frac{1}{D-4} + \frac{1}{2}\left(\gamma + \log\left(\frac{\Lambda^2}{\Lambda'^2}\right)\right)\right]\times$$

$$\times\left(\hat{R}^{;\nu}_{;\mu}+\frac{1}{2}\delta^{\nu}_{\mu}\hat{\nabla}^{\lambda}_{\lambda}-3\hat{\nabla}^{\lambda}_{\lambda}\hat{R}^{\nu}_{\mu}+5\left[\hat{R};\mu^{;\nu}-\delta^{\nu}_{\mu}\hat{\nabla}^{\lambda}_{\lambda}\hat{R}\right]\right)\ , \tag{28}$$

where Ricci tensor and scalar curvature must be taken in the linear approximation with respect to $\gamma_{\mu\nu}$.

Renormalized stress-energy tensor is $\langle T^{\nu}_{\mu}\rangle^{ren}=\lim_{D\to 4}\left(\langle T^{\nu}_{\mu}\rangle^{reg}-\langle T^{\nu}_{\mu}\rangle^{div}\right)$. In the case of monopole and after corresponding rescaling of the scale Λ $\langle T^{\nu}_{\mu}\rangle^{ren}_{Einst}$ can be presented in the form

$$\langle T^{\nu}_{\mu}\rangle^{ren}_{Einst}=\frac{G\eta^2}{2\pi r^4}\left[A^{\nu}_{\mu}+B^{\nu}_{\mu}\log\Lambda r\right]\ . \tag{29}$$

For more details see [10].

As for the renormalized contribution from the bulk, it can be shown that this part has the form

$$\langle T^{\nu}_{\mu}\rangle^{ren}_{bulk}=-\frac{\pi G_5\eta^2}{360}\int\frac{d^3q}{(2\pi)^3}e^{iqx}\frac{K_0\left(|q|/k\right)}{K_1(|q|/k)}|q|^2\log\left(\frac{|q|}{\Lambda}\right)\left[3\sum_{i=1}^{2}\delta_{i\mu}\delta^{\nu}_{i}-2\delta^{\nu}_{\mu}-\frac{q_\mu q^\nu}{q^2}\right]\ . \tag{30}$$

From the last equation we can obtain that at large distances contribution from the bulk is of the order $(kr)^{-2}$ compared to (29), while at $r\ll k^{-1}$

$$\langle T^{0}_{0}\rangle^{ren}_{bulk}=\frac{G_4\eta^2}{120kr^5}\ . \tag{31}$$

Analogous calculations performed for the case of a brane string [4] enable us to show, that

$$\langle T^{0}_{0}\rangle^{ren}_{Einst}=-\frac{G_4\mu}{10\pi^2r^4}\ . \tag{32}$$

At the same time contribution from the bulk is negligible compared to (32) when $r\gg k^{-1}$

$$\langle T^{0}_{0}\rangle^{ren}_{bulk}=-\frac{4G_4\mu}{45\pi^2}\frac{1}{k^2r^6}\log(2k\Lambda^{'}r^2)\ , \tag{33}$$

while near the string, at $r\ll k^{-1}$,

$$\langle T^{0}_{0}\rangle^{ren}_{bulk}=-\frac{G_4\mu}{80\pi^2}\frac{1}{kr^5}\log(r\Lambda^{''})\ . \tag{34}$$

In Egs.(33) and (34) we included in the scales $\Lambda^{'}$ and $\Lambda^{''}$ the terms $k^{-2}r^{-6}$ and $k^{-1}r^{-5}$ correspondingly.

We see that in distinction to the case of the $4D$ Einstein gravity the expression for the total vacuum averaged $\langle T_{\mu\nu}\rangle^{ren}_{vac}$ for both string and monopole space-times contains the Λ-dependent terms which appear due to the renormalization procedure. From the first glance the loss of uniqueness must lead to some problems, because the ordinary approach to the back reaction problem consists of solving the semiclassical Einstein equations with $\langle T_{\mu\nu}\rangle^{ren}$ in the right-hand side. But it can be shown that any variation of Λ may be absorbed by the coefficients in front of the fourth-order terms proportional to ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ appearing in the left-hand side of the $4D$ one-loop Einstein equations (see for example [9])

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\epsilon_1{}^{(1)}H_{\mu\nu}+\epsilon_2{}^{(2)}H_{\mu\nu}=8\pi G\langle T_{\mu\nu}\rangle^{ren}\ . \tag{35}$$

The corresponding renormalization group equations read

$$\Lambda \frac{d\epsilon_1}{d\Lambda} = \frac{G_4}{120\pi} , \quad \Lambda \frac{d\epsilon_2}{d\Lambda} = \frac{G_4}{60\pi} . \tag{36}$$

The exact dependence of $\epsilon_{1,2}$ on Λ should be fixed experimentally. Analogous question was considered for the case of a global monopole in the four-dimensional Einstein's theory in Ref. [10].

4 Conclusions

In this paper we consider gravitational and gravity-induced properties of topological defects embedded in a brane of the Randall-Sundrum infinite brane world. We restrict our consideration to the cases of local string and global monopole which are of the most interest from the cosmological point of view. It is shown that possibility of gravity to propagate in extra dimension changes significantly gravitational field near defects. In particular, corrections from the bulk give the string and monopole an attractive potential which becomes significant near the core. Then we reexamine vacuum polarization effects outside the defects. It is shown, that contribution from the bulk to $\langle T_{\mu\nu} \rangle^{ren}$ tends to zero as $kr \to \infty$, but it becomes significant at short distances.

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