Complete $O(\alpha)$ solution of the μ–decay problem

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Abstract

In this talk I report on the results of a complete $O(\alpha)$ calculation of leptonic μ–decay. The calculation is complete in the sense that all polarization and mass effects have been included in the radiative corrections. I mostly concentrate on the longitudinal polarization of the electron which considerably differs from the naive value $P_e^l = -1$ in the threshold region, both for the Born term and more so for the radiatively corrected case. I also discuss the role of the $O(\alpha)$ anomalous spin–flip contribution and its description in terms of the equivalent particle approach which survives the $m_e \to 0$ limit. Finally, I provide a brief account of the history of the $O(\alpha)$ radiative corrections to leptonic μ–decay. I trace the error done in the first (wrong) 1956 calculation of Behrends, Finkelstein and Sirlin. My account of this historical error differs from that recently given in a talk by Kinoshita on the occasion of the 70th birthday of Sirlin.

1 Introduction

Let me first clarify what I mean by "complete" $O(\alpha)$ solution of the μ–decay problem. With polarization and mass effects included, the decay $\mu^-(\uparrow) \to e^-(\uparrow) + \nu_\mu + \bar{\nu}_e$ is described by five structure functions $G_1, .., G_5$. Here I have disregarded a sixth T–odd structure function G_6 which is zero in the Standard Model. The five structure functions are associated with the rate (G_1), single spin effects (G_2 for the muon, G_3 for the electron) and spin-spin correlation effects (G_4,G_5). The complete $O(\alpha)$ solution of the μ^-–decay problem consists in calculating all five structure functions at next-to-leading order (NLO) in differential and integrated form keeping $m_e \neq 0$.

In most previous calculations the strategy of calculating the radiative corrections to the five structure functions was characterized by the statement "put the electron mass to zero whenever possible", keeping the mass dependence only in logarithmic factors involving $\ln(m_e/m_\mu)$. This holds true for the first correct calculation of the unpolarized spectrum function G_1 given in [1]. The corresponding $m_e \neq 0$ result can be extracted with a little bit of labour from the papers of [2, 3].

Putting the electron mass to zero (whenever possible) is really quite adequate for most of the electron spectrum since the ratio (m_e/m_μ) is quite small. This is no longer true for the threshold region where electron mass effects become important for β–values smaller than $\beta \approx 0.995$. This will be discussed later on when we calculate the longitudinal polarization of the electron. Moreover, after the discovery of the τ–lepton it was not so clear to what extent the mass of the final state lepton could be neglected since in the case $\tau^- \to \mu^- + \nu_\tau + \bar{\nu}_\mu$ the mass ratio (m_μ/m_τ) is not really very small.

Historically, the program of calculating the $O(\alpha)$ corrections to the five structure functions proceeded in several steps. Here one has to distinguish between results for the differential energy distributions which are characterized by the spectrum functions G_i, and the fully integrated results which are characterized by the rate functions $\hat{G}_i = \int \beta x G_i dx$.

Concerning the spectrum functions, one can extract the $m_e \neq 0$ $O(\alpha)$ radiative corrections to the spectrum function G_1 from [2, 3] as remarked on before. The authors of [3] also give results for the spectrum function G_2, however, for $m_e = 0$. In [4] Fischer and Scheck calculated the $m_e = 0$ radiative corrections to G_1, G_2, G_3 and G_4 where one should mention that G_5 describing the transverse polarization of the electron vanishes in the limit $m_e \to 0$. $m_e \neq 0$ results for the polarized spectrum functions G_2 and for G_2, G_3, G_4, G_5 can be found in [5] and in [6], respectively.

Results for the $m_e \neq 0$ integrated rate function $\hat{G}_1$ were first obtained by Nir in the context of semileptonic heavy quark decays [7]. He used a different route to obtain the integrated rate. He started with the differential q^2-distribution calculated in [8] and then obtained the total rate by q^2–integration. Because semileptonic heavy quark decays have a structure similar to leptonic μ^-–decays in the Standard Model his results apply also to leptonic μ–decays. The $m_e \neq 0$ polarized rate functions $\hat{G}_2$ and $\hat{G}_2, \hat{G}_3, \hat{G}_4, \hat{G}_5$ were again obtained by Arbuzov [5] and by us [6], respectively. It is not quite clear what caused the long delay from 1958 to 2003 to complete the missing $O(\alpha)$ pieces in leptonic μ–decays. Part of the reason is that the necessary calculations are quite involved and therefore had to wait for the advent of modern computers with their symbolic computation facilities.

For us calculating the complete $m_e \neq 0$ $O(\alpha)$ radiative corrections to μ–decay was a natural outgrowth of corresponding $O(\alpha_s)$ polarization calculations that we had done for the semileptonic quark decays $t \to b + l^+ + \nu_l$ and $b \to c + l^- + \bar{\nu}_l$ [9] which, in the Standard Model, have a structure similar to leptonic μ–decay. These calculations had been done keeping the full mass dependence of the final state quark. This is important in particular for the case $b \to c + l^- + \bar{\nu}_l$ since, in this case, the mass of the final state charm quark mass can certainly not be neglected. The expertise in handling the full mass dependence in semileptonic heavy quark decays was then exported to leptonic μ–decays.

2 Angular decay distribution

The angular decay distribution of the semileptonic decay of a polarized muon into a polarized electron is given by [6]

$$\begin{aligned} \frac{d\Gamma}{dx\, d\cos\theta_P} &= \beta x\, \Gamma_0 (G_1 + G_2 P \cos\theta_P + G_3 \cos\theta + G_4 P \cos\theta_P \cos\theta \\ &+ G_5 P \sin\theta_P \sin\theta \cos\chi + G_6 P \sin\theta_P \sin\theta \sin\chi)\ , \end{aligned} \tag{1}$$

where θ_P is the polar angle between the electron and the polarization vector $\vec{P}$ of the μ^- in the μ^- rest system , and θ and χ are the polar and azimuthal angles describing the orientation of the spin quantization axis of the electron [6]. The Born term rate for vanishing electron mass $m_e = 0$ is given by $\Gamma_0 = G_F^2 m_\mu^5 / 192\pi^3$. As usual I have defined a scaled electron energy $x = 2E_e/m_\mu$. Further, the velocity of the electron is determined by $\beta = |\vec{p_e}|/E_e = \sqrt{1 - (4y^2/x^2)}$ where $y = m_e/m_\mu$. As mentioned before,

G_1 is the unpolarized spectrum function, G_2 and G_3 are single spin polarized spectrum functions referring to the spins of the μ^- and e^-, resp., and G_4, G_5 and G_6 describe spin–spin correlations between the spin vectors of the muon and electron. G_6 represents a so–called T–odd observable which is identically zero in the Standard Model [6]. As mentioned before, the structure function G_5 is proportional to the electron mass and therefore vanishes for $m_e \to 0$.

3 Longitudinal polarization of the electron

Here I concentrate on the longitudinal polarization of the electron which, for an unpolarized muon, is given by

$$P_e^l(x) \;=\; \frac{G_3(x)}{G_1(x)}. \tag{2}$$

For the Born term contribution one obtains

$$P_e^l(x) = -\beta \frac{x(3-2x+y^2)}{x(3-2x)-(4-3x)y^2} \quad . \tag{3}$$

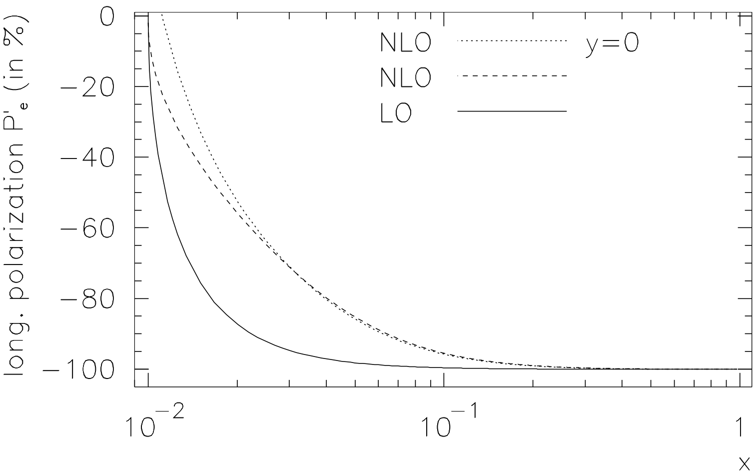

Figure 1: Longitudinal polarization of the electron in leptonic muon decays at LO (full line) and NLO (dashed line) as a function of the scaled energy $x = 2E_e/m_\mu$ [6]. The NLO (dotted line) curve corresponds to keeping $m_e \neq 0$ in the Born contribution and $m_e = 0$ in the $O(\alpha)$ contribution.

The LO curve starts to deviate from the naive result $P_e^l = -1$ at around $x = 0.1$ ($\beta = 0.995$) where mass effects start setting in. It is curious to note that the LO curve is very well described by the functional behaviour $P_e^l = -\beta$. The correction to the approximate result $P_e^l = -\beta$ is of $O(1\%)$ or less such that the correct and approximate LO curves are not discernible at the scale of Fig. 1. I make mention of this fact since in some recent text books the result $P_e^l = -\beta$ has been claimed to be an exact result for left-chiral heavy fermions [10]. Eq.(3) shows that $P_e^l = -\beta$ cannot be an exact result.

Numerically, however, $P_e^l = -\beta$ appears to be a very good approximation, at least in this application.

The radiative corrections to the polarization of the electron become quite pronounced starting at $x = 0.1$ ($\beta = 0.995$). We also show an approximate NLO curve where $m_e \neq 0$ for the Born term and $m_e = 0$ for the $O(\alpha_s)$ contribution as suggested by [4]. The approximate NLO curve does not have the correct threshold behaviour $P_e^l \simeq -\beta$ near threshold at $x = 2y \simeq 0.00967$ resulting from the wrong threshold behaviour of the radiative corrections which starts showing up at $x = 0.02$ ($\beta = 0.875$).

4 Calculation of the tree graph contribution

The NLO charge-side tensor describing the $\mu^- \to e^- + \gamma$ tree-graph transition in charge retention form is given by [6]

$$\begin{aligned} C^{\alpha\beta} &= \sum_{\gamma-\mathrm{spin}} \mathcal{M}^\alpha \mathcal{M}^{\beta\dagger} = \frac{e^2}{2}\Big\{ \frac{1}{k\cdot p_e}\Big(\frac{k\cdot\bar p_e - m_e^2}{k\cdot p_e} + \frac{p_\mu\cdot\bar p_e}{k\cdot p_\mu}\Big)(k^\alpha \bar p_\mu^\beta + k^\beta \bar p_\mu^\alpha - k\cdot\bar p_\mu g^{\alpha\beta}) \\ &+ \frac{1}{k\cdot p_\mu}\Big(\frac{k\cdot\bar p_\mu + m_\mu^2}{k\cdot p_\mu} - \frac{p_e\cdot\bar p_\mu}{k\cdot p_e}\Big)(k^\alpha \bar p_e^\beta + k^\beta \bar p_e^\alpha - k\cdot\bar p_e g^{\alpha\beta}) \\ &+ \frac{k\cdot\bar p_e}{(k\cdot p_e)^2}(p_e^\alpha \bar p_\mu^\beta + p_e^\beta \bar p_\mu^\alpha - p_e\cdot\bar p_\mu g^{\alpha\beta}) - \frac{k\cdot\bar p_\mu}{(k\cdot p_\mu)^2}(p_\mu^\alpha \bar p_e^\beta + p_\mu^\beta \bar p_e^\alpha - p_\mu\cdot\bar p_e g^{\alpha\beta}) \\ &+ \frac{k\cdot\bar p_\mu}{(k\cdot p_e)(k\cdot p_\mu)}(p_e^\alpha \bar p_e^\beta + p_e^\beta \bar p_e^\alpha - m_e^2 g^{\alpha\beta}) - \frac{k\cdot\bar p_e}{(k\cdot p_e)(k\cdot p_\mu)}(p_\mu^\alpha \bar p_\mu^\beta + p_\mu^\beta \bar p_\mu^\alpha - m_\mu^2 g^{\alpha\beta})\Big\} \\ &- \frac{e^2}{2}\Big(\frac{m_\mu^2}{(k\cdot p_\mu)^2} + \frac{m_e^2}{(k\cdot p_e)^2} - \frac{2 p_e\cdot p_\mu}{(k\cdot p_e)(k\cdot p_\mu)}\Big)(\bar p_e^\alpha \bar p_\mu^\beta + \bar p_e^\beta \bar p_\mu^\alpha - \bar p_e\cdot\bar p_\mu g^{\alpha\beta})\ . \end{aligned} \tag{4}$$

The momentum of the radiated photon is denoted by k. I have ommitted an $\alpha \leftrightarrow \beta$ antisymmetric piece in Eq.(4) for the reason that it does not contribute to the differential electron spectrum.

Eq.(4) is written in a very compact way. First, polarization effects are included by making use of the very compact notation

$$\bar p_\mu^\alpha = p_\mu^\alpha - m_\mu s_\mu^\alpha\ , \qquad \bar p_e^\alpha = p_e^\alpha - m_e s_e^\alpha\ , \tag{5}$$

where s_μ^α and s_e^α are the polarization four–vectors of the μ^- and e^-. Second, in the last line of (4) I have isolated the infrared singular piece of the charge–side tensor which is given by the usual soft photon factor multiplying the Born term contribution. Technically this is done by writing

$$C^{(\alpha)\alpha\beta} = \Big(C^{(\alpha)\alpha\beta} - C^{(\alpha)\alpha\beta}(\text{soft photon})\Big) + C^{(\alpha)\alpha\beta}(\text{soft photon})\ . \tag{6}$$

The remaining part of the charge–side tensor in (4) is referred to as the hard photon contribution. It is infrared finite and can thus be integrated without a regulator photon mass. The integration of the infrared singular piece with a regulator photon mass will be discussed in the next section including a discussion of the errors that had been made in the first evaluation of the infrared contribution [2].

5 Historical note on soft photon regularization

Let me now turn to the calculation of the soft photon contribution. The soft photon (s.ph.) transition matrix element reads

$$M^{\alpha}_{s.ph.} = (\frac{p^{\alpha}_{\mu}}{p_{\mu}k} - \frac{p^{\alpha}_{e}}{p_{e}k}) \cdot \tag{7}$$

The integration over the relevant phase space is determined by

$$I = \frac{1}{4\pi}\int_{-1}^{1} dz \int_{0}^{k_{max}(z)} \frac{d^3k}{k_0}|M_{s.ph.}|^2, \tag{8}$$

where $z = \cos\theta$ is the cosine of the angle between the electron and the photon, and $k_{max}(z)$ is the maximal photon momentum $k_{max}(z) = m_{\mu}(1 + y^2 - x)/(2 - x(1 - \beta z))$.

The squared soft photon matrix element in (8) is given by

$$|M_{s.ph.}|^2 = \sum_{\lambda=\pm 1} M^{\alpha}_{s.ph.} M^{*\beta}_{s.ph.} \epsilon^{*}_{\alpha}(\lambda)\epsilon_{\beta}(\lambda)\,. \tag{9}$$

The spin summed product of polarization vectors in (9) can be replaced by the metric tensor as follows

$$\sum_{\lambda=\pm 1} \epsilon^{*\alpha}(\lambda)\epsilon^{\beta}(\lambda) = \mathrm{diag}(0,1,1,0) \;\rightarrow\; \mathrm{diag}(-1,1,1,1) = -g^{\alpha\beta}. \tag{10}$$

This corresponds to using the Feynman gauge which was also used to calculate Eq.(4). The longitudinal and scalar pieces can be added to the spin sum in (10) since their contributions in (9) cancel due to gauge invariance. This can be explicitly checked by evaluating the longitudinal and scalar pieces in the rest system of the μ^- with the z–axis along the photon direction, where $k^{\mu} = (k_0; 0, 0, k)$, $p^{\alpha}_{\mu} = (m_{\mu}; 0, 0, 0)$ and $p_e = (E_e; |\vec{p_e}|\sin\theta, 0, |\vec{p_e}|\cos\theta)$. For the squared soft photon matrix element the replacement in (10) leads to

$$|M_{s.ph.}|^2_T = \beta^2 \frac{1 - z^2}{(k_0 - \beta k z)^2} \;\rightarrow\; |M_{s.ph.}|^2 = \beta^2 \frac{k_0^2 - k^2 z^2}{k_0^2(k_0 - \beta k z)^2} \cdot \tag{11}$$

It is instructive to split the squared soft photon matrix element into its transverse and longitudinal/scalar part. One obtains

$$|M_{s.ph.}|^2 = |M_{s.ph.}|^2_T + |M_{s.ph.}|^2_C = \beta^2\Big(\frac{1 - z^2}{(k_0 - \beta k z)^2} + (k_0^2 - k^2)\frac{z^2}{k_0^2(k_0 - \beta k z)^2}\Big) \cdot \tag{12}$$

The longitudinal and scalar contributions in $|M_{s.ph.}|^2_C$ are proportional to k_0^2 and k^2. For on-shell photons with $k_0 = k$ the longitudinal and scalar contributions cancel as asserted before.

After doing the azimuthal integration the soft photon factor (8) is given by

$$I = \frac{\beta^2}{2} \int_{-1}^{1} dz \int_0^{k_{max(z)}} dk \frac{k^2}{k_0^3} \frac{k_0^2 - k^2 z^2}{(k_0 - \beta k z)^2} \; . \tag{13}$$

Similar to Eq.(12) it is instructive to split the integral into a transverse contribution I_T and a longitudinal/scalar contribution I_C. One obtains

$$I := I_T + I_C = \frac{\beta^2}{2} \int_{-1}^{1} dz \int_0^{k_{max(z)}} dk \, \Big(\frac{k^2}{k_0} \, \frac{1 - z^2}{(k_0 - \beta k z)^2} \; + \; (k_0^2 - k^2) \, \frac{k^2}{k_0^3} \, \frac{z^2}{(k_0 - \beta k z)^2} \Big) \; . \tag{14}$$

At this point we introduce a (small) photon regulator mass m_γ through $k_0^2 - k^2 = m_\gamma^2$. The regulator mass m_γ is used to regularize the infrared singularity present in Eqs.(13) and (14). One can still use the metric tensor in (10) since the scalar piece $k^\alpha k^\beta / m_\gamma^2$ in the spin 1 propagator gives zero when contracted with the square of the soft photon matrix element $M^\alpha_{s.ph.} M^{*\beta}_{s.ph.}$. Technically, the integration is best done by changing to the variable t via $k = m_\gamma (t^2 - 1)/2t$ in order to get rid of the square root factor in the photon energy $k_0 = \sqrt{k^2 + m_\gamma^2}$. MATHEMATICA will do the rest for you.

For reasons of brevity I only list the $y \to 0$ results of the two integrations in (14). One has ($\Lambda = m_\gamma / m_\mu$)

$$I_T = 2 \ln \frac{1 - x}{\Lambda} \Big(\ln(\frac{x}{y}) - 1 \Big) + Li_2(x) - \ln^2(\frac{x}{y}) + \frac{\pi^2}{12} + (1 - \frac{1}{x}) \ln(1 - x) - 1 \; , \tag{15}$$

and

$$I_C = \ln(\frac{x}{y}) + 1 - \frac{\pi^2}{4} \cdot \tag{16}$$

For the sum of the two contributions one obtains

$$I_T + I_C = \Big(\ln(\frac{x}{y}) - 1 \Big) \Big(2 \ln \frac{(1 - x)}{\Lambda} - \ln(\frac{x}{y}) \Big) + Li_2(x) - \frac{\pi^2}{6} + (1 - \frac{1}{x}) \ln(1 - x) \, . \tag{17}$$

Behrends, Finkelstein and Sirlin in their 1956 paper [2] set $k_0 = k$ in the integrand of the transverse contribution of (14). Of course, setting $k_0 = k$ in the total contribution (13) gives the same result in agreement with the arguments presented before. They then obtain (using their notation)

$$\begin{aligned} V &= \frac{\beta^2}{2} \int_{-1}^{1} dz \int_0^{k_{max(z)}} dk_0 \frac{1}{k_0} \frac{1 - z^2}{(1 - \beta z)^2} = \frac{\beta^2}{2} \int_{-1}^{1} dz \ln \frac{k_{max}(z)}{m_\gamma} \frac{1 - z^2}{(1 - \beta z)^2} \\ &= \ln \frac{(1 - x)}{2\Lambda} \Big(\ln(\frac{x}{y}) - 1 \Big) + Li_2(x) + (1 - \frac{1}{x}) \ln(1 - x) - 1 \; . \end{aligned} \tag{18}$$

It is obvious that the result (18) differs from the true transverse contribution (15). Apparently, Behrends, Finkelstein and Sirlin committed two mistakes in their 1956 paper [2]. First, they considered only the transverse contribution of the massive photon instead of including also its longitudinal part. Second, they did not correctly calculate the transverse contribution by erraneously setting $k_0 = k$ in the matrix element. Kinoshita in a recent talk given on the occasion of the 70th birthday of Sirlin identifies only the first mistake [11].

The two mistakes were corrected in the 1959 paper [3] of Kinoshita and Sirlin by adding a compensation term C (their notation) to their 1956 result, where

$$C = I_T + I_C - V = \Big(\ln(\frac{x}{y}) - 1\Big)\Big(2\ln 2 - \ln(\frac{x}{y})\Big) + 1 - \frac{\pi^2}{6}\,. \tag{19}$$

Why so much ado about a mistake done many years ago? Well, the mistake is in some sense historical since the 1956 result violates the so-called Kinoshita–Lee–Nauenberg theorem formulated a few years later on [12, 13]. The Kinoshita–Lee–Nauenberg theorem states that the integrated rate should not contain any logarithmic dependence on the electron mass. However, when integrating the 1956 result one finds

$$\Gamma_{(1956)} = \Gamma_0 \Big(1 + \frac{\alpha}{2\pi}\Big(\frac{25}{4} - \pi^2 + 4\,\Big[\,\ln^2 y + (3 + 2\ln 2)\ln y + 2 + 4\ln 2 + \frac{\pi^2}{6}\,\Big]\Big)\Big)\,, \tag{20}$$

which clearly violates the Kinoshita–Lee–Nauenberg theorem. The correct result is obtained by setting the square bracket in (20) to zero which shows that the correct result does in fact satisfy the Kinoshita–Lee–Nauenberg theorem. In fact, the observation that there is no logarithmic mass dependence in the leptonic μ decay rate was very likely a progenitor of the celebrated Kinoshita–Lee–Nauenberg theorem.

6 The anomalous spin-flip contribution

In this section we concentrate on one interesting aspect of the μ–decay problem, namely on the so–called anomalous spin-flip contribution which, even in the chiral limit, flips the helicity of the final–state lepton at NLO.

Collinear photon emission from a massless fermion line can flip the helicity of the massless fermion contrary to naive expectation. This has been discussed in a variety of physical contexts (see e.g. references in [6]). This is a "m_e/m_e" effect where the m_e in the numerator is a spin flip factor and the m_e in the denominator arises from the collinear configuration. In the limit $m_e \to 0$ the helicity flip contribution survives whereas it is not seen in massless QED.

We shall discuss this phenomenon in the context of the left–chiral $\mu \to e$ transition. At the Born term level an electron emerging from a weak $(V - A)$ vertex is purely left–handed in the limit $m_e = 0$. Naively, one would expect this to be true also at $O(\alpha)$ or at any order in α because in massless QED photon emission from the electron is helicity conserving.

Let us take a closer look at the anomalous helicity flip contribution in leptonic $\mu \to e$ decays by considering the unnormalized density matrix element ρ_{++} of the final state electron which is obtained by setting $\cos\theta = 1$ in (1) (remember that G_5 vanishes for $m_e \to 0$, and $G_6 = 0$ in the Standard Model). One has

$$\frac{d\Gamma^{(++)}}{dx} = \beta x\,\Gamma_0\Big(G_1 + G_3\Big)\,. \tag{21}$$

Contrary to naive expectations one finds non–vanishing right–handed (++) contributions which survive the $m_e \to 0$ limit when one takes the $m_e \to 0$ limit of the NLO contributions to (21) [6]. In fact, one finds

$$\frac{d\Gamma^{(++)}}{dx} = \frac{\alpha}{6\pi}\Gamma_0\Big((1-x)^2(5-2x)\Big) \,. \tag{22}$$

The result is rather simple. In particular, it does not contain any logarithms or dilogarithms. The simplicity of the right–handed contribution becomes manifest in the equivalent particle description of μ–decay where, in the peaking approximation, μ–decay is described by the two–stage process $\mu^- \to e^-$ followed by the branching process $e^- \to e^- + \gamma$ characterized by universal splitting functions $D_{nf/hf}(z)$ [14]. The symbols nf and hf stand for a helicity non-flip and helicity flip of the helicity of the electron. In the splitting process z is the fractional energy of the emitted photon. The off–shell electron in the propagator is replaced by an equivalent on–shell electron in the intermediate state. Since the helicity flip contribution arises entirely from the collinear configuration it can be calculated in its entirety using the equivalent particle description.

The helicity flip splitting function is given by $D_{hf}(z) = \alpha z/(2\pi)$, where $z = k_0/E' = (E'-E)/E' = 1 - x/x'$, and where k_0 is the energy of the emitted photon. E' and E denote the energies of the initial and final electron in the splitting process. The helicity flip splitting function has to be folded with the appropriate $m_e = 0$ Born term contribution. The lower limit of the folding integration is determined by the soft photon point where $E' = E$. The upper limit is determined by the maximal energy of the initial electron $E' = m_\mu/2$. One obtains

$$\begin{aligned}
\frac{d\Gamma^{(++)}}{dx} &= \int_0^1 dx' \int_0^1 dz \frac{d\Gamma^{\rm Born}(x')}{dx'} D_{hf}(z)\delta(x - x'(1-z)) \\
&= \frac{\alpha}{2\pi}\int_x^1 dx' \frac{1}{x'}\frac{d\Gamma^{\rm Born}(x')}{dx'}(1 - \frac{x}{x'}) \\
&= \frac{\alpha}{\pi}\Gamma_0 \int_x^1 dx'(x'-x)\Big(3-2x'\Big) \\
&= \frac{\alpha}{6\pi}\Gamma_0\Big((1-x)^2(5-2x)\Big) \,,
\end{aligned} \tag{23}$$

where $d\Gamma^{\rm Born}(x')/dx' = \Gamma_0 2x'^2(3-2x')$, and $\delta(x - x'(1-z)) = \delta(z - \frac{x'-x}{x'})/x'$. The δ–function expresses energy conservation in the splitting process. The integration over z shifts the lower boundary of the x' integration from 0 to x because of the δ–function . The final result exactly reproduces the result (22).

Numerically, the flip spectrum function is rather small compared to the $O(\alpha)$ no-flip spectrum function. However, when averaging over the spectrum the ratio of the $O(\alpha)$ flip and no-flip contributions amounts to a non-negligible (-12%), due to cancellation effects in the $O(\alpha)$ no-flip contribution. It is clear that the corresponding helicity flip effect is larger in QCD due to $\alpha_s/\alpha \approx 10$.

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