Quantum fluctuations of chiral condensate in the analytically regularized Nambu–Jona-Lasinio model

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Abstract

The problem of quantum fluctuations of the chiral condensate due to next-to-leading contributions is investigated. Meson contributions to the chiral condensate are calculated in the Nambu–Jona-Lasinio model with the analytical (dimensional) regularization in the framework of the mean-field expansion. The pion contribution can be significant, while the sigma-meson contribution is small in the regime of physical values of the model parameters. Non-pole contributions are also estimated and found to be small.

Introduction and Summary

The Nambu–Jona-Lasinio model (NJL) is a successful effective model of quantum chromodynamics of light hadrons in the non-perturbative region (See [1, 2] for reviews). In the majority of investigations the NJl model have been considered in the mean-field approximation. The success in phenomenological applications has stimulated an analysis of the structure of the NJL model beyond the mean-field approximation, i.e., in the next-to-leading orders of mean-field expansion. Such analysis is necessary for a clarification of the region of applicability of results and its stability with regard to a variation of parameters and quantum fluctuations due to higher-order effects.

In this talk I report some results of the investigation of the NJL model with the dimensional (analytical) regularization in the next-to-leading order of the mean-field expansion.

Due to non-renormalizability a regularization is an essential aspect of the NJL model. In contrast to renormalized models, a parameter of regularization in the NJL model enters the physical quantities and it is one of essential parameters of the model. For this reason the least common regularization for NJL model is the dimensional regularization, since the parameter of dimensional regularization traditionally is treated as a deviation from the physical dimension of a space and does not permit any physical interpretation in this treatment. However, an alternative treatment of the dimensional regularization exists – as a variant of an analytical regularization. In this treatment all calculations are made in the four-dimensional Euclidean momentum space, and the regularization parameter is treated as a power of a weight function which regularizes divergent integrals. Such treatment of the dimensional regularization have been developed and applied to the NJL model in the mean-field approximation by Krewald and Nakayama [3].

A possible treatment of the parameter of this analytical regularization is some power of a gluon influence on the effective four-fermion quark self-action of the NJL model.

Section 1 is based on results of R. Jafarov and myself [4] and is devoted to calculations of meson contributions to the quark chiral condensate of the NJL model. The pion contribution is significant, and at a bound of admissible values of the model parameters it tends to infinity, i.e., the model is unstable with respect to quantum fluctuations near this bound. For physical values of the parameters this contribution is $10 \div 20\%$ of the leading contribution. The sigma-meson contribution is small for admissible values of the parameters.

In Section 2 non-pole contributions to the chiral condensate are estimated. These contributions are found to be small. Also in this section the interesting phenomenon of cancellation in the scalar amplitude is discussed: at some value of the analytic regularization parameter almost all contributions to the scalar amplitude are canceled. This effect leads to a "disappearance" of the sigma-meson from the spectrum of the analytically regularized NJL model.

Quite briefly our result can be formulated as a following statement: the analytically regularized NJL model does not contain any pathological quantum fluctuations connected with the scalar-meson contributions for physical values of parameters, though the pion contribution is significant and should be taken into account at phenomenological treatments of the NJL-type models.

1 Analytically regularized NJL model. A mean-field expansion and scalar meson contributions to the chiral condensate

We consider SU(2) NJL model, which is a theory of a spinor field $\psi(x)$ with two flavors (isospin), n_c colors, and $SU_V(2) \times SU_A(2)$-invariant self-action

$$\mathcal{L}_{int} = \frac{g}{2}\Big[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau^a\psi)^2\Big].$$

Here τ^a are Pauli matrices.

Quark propagator $S^{(0)}$ in the leading mean-field approximation (Hartree approximation) is

$$S^{(0)} = \frac{1}{m - \hat{p}},$$

where the dynamical mass m is a solution of gap equation

$$1 = -8ign_c \int \frac{d\tilde{p}}{m^2 - p^2}. \tag{1}$$

Here and below $d\tilde{p} \equiv d^4p/(2\pi)^4$. The divergent integral in r.h.s. should be considered with some regularization.

In the leading approximation the chiral condensate is

$$\chi^{(0)} = i\,\mathrm{tr}\,S^{(0)}(0) = -\frac{m}{g}. \tag{2}$$

The two-particle amplitude in the mean-field approximation is

$$A = 1 \otimes 1 A_\sigma + \tau^a \otimes \tau^a A_\pi \tag{3}$$

Here scalar amplitude A_σ and pseudoscalar amplitude A_π are

$$A_\sigma = -\frac{ig}{1 - L_S}, \quad A_\pi = \frac{ig}{1 + L_P}$$

where L_S and L_P are scalar and pseudoscalar fermion loops.

Taking into account gap equation (1) one can obtain the following representations for A_σ and A_π in momentum space:

$$A_\sigma(p) = \frac{1}{4n_c(4m^2 - p^2)I_0(p^2)} \tag{4}$$

$$A_\pi(p) = \frac{1}{4n_c p^2 I_0(p^2)}. \tag{5}$$

Here

$$I_0(p^2) = \int d\tilde{q} \frac{1}{(m^2 - (p+q)^2)(m^2 - q^2)}. \tag{6}$$

Meson contributions to the chiral condensate can be calculated in the next-to-leading term of the mean-field expansion. A systematic construction of the mean-field expansion can be made by using the bilocal source formalism. For this purpose we shall follow the method of an iterative solution of the Schwinger-Dyson equation for the generating functional of Green functions (see [5] for a brief review of the method and [4] for details of the mean-field expansion in the NJL model).

The next-to-leading correction to the propagator $S^{(1)}$ in the mean-field expansion is defined by the equation for the first-step mass operator $\Sigma^{(1)} = [S^{(0)}]^{-1} \star S^{(1)} \star [S^{(0)}]^{-1}$ (in x-space):

$$\Sigma^{(1)}(x) = ig\delta(x)\,\mathrm{tr}\, S^{(1)}(0) + S^{(0)}(x)A_\sigma(x) + 3S^{(0)}(-x)A_\pi(x). \tag{7}$$

Due to the non-renormalizability, the NJL model should be regularized, and the regularization is an essential component of the model. We shall use some special variant of the dimensional regularization, which was proposed in the work [3]. In this approach the dimensional regularization is, essentially, a variant of the analytical regularization.

Let us consider the approach for the gap equation of the NJL model as an example. The gap equation in the Euclidean space after an angle integration is

$$1 = 4gn_c \frac{\Omega_4}{(2\pi)^4} \int \frac{q_e^2 dq^2}{m^2 + q_e^2}.$$

In correspondence with the prescriptions of [3] we modify the integrand by the weight function

$$w_D(q_e^2) = \left(\frac{\mu^2}{q_e^2}\right)^{2-D/2}$$

and rescale parameter μ^2 as

$$(\mu^2)^{2-D/2} = \frac{\Omega_D}{\Omega_4} \frac{(2\pi)^4}{(2\pi)^D} (M)^{2-D/2}.$$

where $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$. Then we obtain for the gap equation

$$1 = \kappa \left(\frac{m^2}{4\pi M^2} \right)^{D/2-2} \Gamma(1 - D/2),$$

where dimensionless constant κ is introduced: $\kappa = gn_c m^2/2\pi^2$.

This equation exactly corresponds to the calculation of the initial integral with the formal prescription of D-dimensional integration

$$d\tilde{q} \equiv \frac{d^4 q}{(2\pi)^4} \to \frac{(M^2)^{2-D/2} d^D q}{(2\pi)^D},$$

but in our case the calculation was performed in the usual 4-dimensional space, i.e. in our treatment parameter D is not a dimension of some space, but a parameter which provides the convergence. In particular, we are not constrained by the limit $D \to 4$ for the treatment of results.

Below we shall use *regularization parameter* ξ, $\xi = (2 - D)/2$, and the gap equation has the form

$$1 = \kappa \Gamma(\xi) \left(\frac{4\pi M^2}{m^2} \right)^{1+\xi}. \tag{8}$$

The region of convergence of the integral is $0 < \xi < 1$. As we shall see, there is also a region for the physical values of the model parameters.

The integral I_0 (see (6)) which is a part of the scalar amplitudes A_σ and A_π, also can be calculated with above prescriptions. Going to the Euclidean metric, introducing the standard Feynman parameterization, and translating integration variable (which is possible due to translational invariance of the procedure, see [3]) we can made the angle integration. In correspondence with the rules, we introduce weight function $w_D(q_e^2)$ and, after the same rescaling, obtain the result which also exactly corresponds to the result of integration with the formal transition to D-dimensional space:

$$I_0(p^2) = \int d\tilde{q} \frac{1}{(m^2 - (p+q)^2)(m^2 - q^2)} = \frac{i\xi\Gamma(\xi)}{(4\pi)^2} \int_0^1 du \left(\frac{4\pi M^2}{m^2 - u(1-u)p^2} \right)^{1+\xi}. \tag{9}$$

Taking into account the gap equation (8), we can exclude $\Gamma(\xi)(4\pi M^2)^{1+\xi}$ from (9) and obtain for I_0:

$$I_0(p^2) = \frac{i}{(4\pi)^2} \frac{\xi}{\kappa} \int_0^1 du \left(1 - u(1-u) \frac{p^2}{m^2} \right)^{-1-\xi} = \frac{i}{(4\pi)^2} \frac{\xi}{\kappa} F(1+\xi, 1; 3/2; \frac{p^2}{4m^2}). \tag{10}$$

Here $F(a, b; c; z)$ is the Gauss hypergeometric function.

The pole terms for the scalar amplitudes in the analytical regularization are:

$$A_\sigma^{pole} = \frac{1}{4n_c(4m^2 - p^2) I_0(4m^2)} = \frac{igm^2(1+2\xi)}{(4m^2 - p^2)n_c\xi}, \tag{11}$$

$$A_\pi^{pole} = \frac{1}{4n_c p^2 I_0(0)} = -\frac{igm^2}{p^2 n_c \xi}. \tag{12}$$

For pion decay constant $f_\pi = 93$ MeV we obtain very simple formula

$$f_\pi^2 = \frac{\xi}{2g}. \qquad (13)$$

We shall use also an expression for width $\Gamma_{\pi^0\gamma\gamma} = 7.7$ KeV of decay $\pi^0 \to 2\gamma$ (see [3]). In our notation this formula is

$$\Gamma_{\pi^0\gamma\gamma} = \frac{\alpha^2 m_\pi^3 \xi^2 (1+\xi)^2}{64\pi^3 f_\pi^2 \kappa^2}. \qquad (14)$$

Using these formulas, one can fix parameters m, g(or κ) and ξ of the analytically-regularized NJL model in the leading mean-field approximation.

A new feature of our approach in comparison with that of Krewald and Nakayama [3] is a systematical exploiting of the gap equation to exclude the dependence of the model parameters on the dimensional parameter M, and, correspondingly, a significant simplification of formulas for observable objects.

As a measure of quantum fluctuations of the chiral field, consider a ratio of the first-step condensate $\chi^{(1)} = i \,\mathrm{tr}\, S^{(1)}(0)$ to the leading-approximation condensate $\chi^{(0)}$:

$$r \equiv \frac{\chi^{(1)}}{\chi^{(0)}} = r_\sigma + r_\pi \qquad (15)$$

Here r_σ is a scalar contribution and r_π is a pseudoscalar contribution. The integrals for r_π are calculated in the analytical regularization in a closed form and yield a very simple expression (at $n_c = 3$) :

$$r_\pi = \frac{1}{8\xi}. \qquad (16)$$

Scalar contribution r_σ is also a function of parameter ξ only and can be represented as an integral with the Gauss hypergeometric function (see [4]). Our results indicate, that quantum fluctuations of the chiral condensate in the NJL model can be significant at some values of the regularization parameter ξ. The fluctuations caused by pseudoscalar field are large at $\xi \to 0$. The sigma-meson contribution is small in the region of physical values of the model parameters.

Numerical results are following:

At $c = -160$ MeV we have $\xi \simeq 1$, $m \simeq 475$ MeV, $\kappa \simeq 2$. At this value of ξ the correction to condensate is about 3%, i.e. the model is stable with respect to meson fluctuations. But such a low value of the chiral condensate hardly corresponds to the phenomenology – it lead to large current quark masses.

At $c = -200$ MeV we obtain $\xi \simeq 0.44$, $m \simeq 400$ MeV, $\kappa \simeq 0.62$. The correction to the condensate is 9%.

At $c = -250$ MeV we obtain $\xi \simeq 0.2$, $m \simeq 370$ MeV, $\kappa \simeq 0.24$, and the correction to the condensate is more then 18%.

Here the condensate is $c = (\chi/2)^{1/3}$.

The calculated corrections permit us to modify the choice of parameters by the following modification of the condensate formula

$$\chi = -\frac{m^*}{g^*}[1 + r(\xi^*)]. \qquad (17)$$

This modified choice of parameters give us

– at $c = -200$ MeV: $\xi^* \simeq 0.56$, $m^* \simeq 420$ MeV, $\kappa^* \simeq 0.86$; the condensate correction is 7%;

– at $c = -250$ MeV: $\xi^* \simeq 0.3$, $m^* \simeq 380$ MeV, $\kappa^* \simeq 0.39$; the condensate correction is 13%;

Apparently, the modification decreases the fluctuation of condensate, i.e. it stabilizes the situation.

2 Beyond the pole approximation. Cancellations in the scalar amplitude

The calculations of Section 1 were performed in the pole approximations (11) and (12) for amplitudes A_σ and A_π. The pole approximation (12) for pseudoscalar amplitude A_π is well-defined for the admissible values of regularization parameter ξ. But similar approximation (11) for scalar amplitude A_σ is a questionable thing.

The region of the convergence for integral I_0 is $-1 < \xi < 1$. Hypergeometric function $F(1+\xi, 1; 3/2; z)$ (see eq. (10)) is singular in the point $z = 1$ (i.e., at $p^2 = 4m^2$) at $\xi > -1/2$. This threshold singularity is an artifact of the regularization, and, consequently, we should consider pole approximation (12) of the scalar amplitude as an analytical continuation from the region $-1 < \xi < -1/2$ to the physical region $0 < \xi < 1$. This analytical continuation seems to be necessary for a physical interpretation of the sigma-meson as a particle in the spectrum of the NJL model.

On the other hand, if one considers the scalar amplitudes A_σ and A_π beyond the pole approximation, then an interesting phenomenon in the analytically regularized NJL model arises. At $\xi = 1/2$ the hypergeometric function in eq.(10) reduces to a simple elementary function: $F(3/2, 1; 3/2; z) = (1-z)^{-1}$, and we obtain:

$$I_0 \mid_{\xi=1/2} = \frac{i}{4gn_c(4m^2 - p^2)}, \tag{18}$$

and, in correspondence with eqs. (4) and (5):

$$A_\sigma = -ig, \quad A_\pi = ig - \frac{4igm^2}{p^2}. \tag{19}$$

These extremely simple formulas for the amplitudes demonstrate an almost full cancellation of contributions to scalar amplitude A_σ at $\xi = 1/2$ (except of the leading order of perturbation theory). One of the consequences of such cancellation is a "disappearance" of the sigma-meson from the spectrum of the NJL model.

The formulas (19) for the amplitudes enable us also to estimate the corrections to the chiral condensate beyond the pole approximation. At $\xi = 1/2$ by using amplitudes (19) we obtain

$$r_\sigma = 0, \quad r_\pi = \frac{1}{4}. \tag{20}$$

Apparently the pseudoscalar contribution is exactly the same as the pole contribution of eq.(16). Hence, non-pole contributions at $\xi = 1/2$ disappear: $r_\sigma^{non-pole} = r_\pi^{non-pole} = 0$.

This exact result gives us a possibility to estimate the non-pole contributions near this value of regularization parameter ξ (i.e., in the whole region of physical values of the model parameters) as a small one.

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