Analysis and verification of relativistic wave equations for spin-1 particles

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Abstract

The Foldy-Wouthuysen transformation for relativistic spin-1 particles interacting with nonuniform electric and uniform magnetic fields is performed. The Hamilton operator in the Foldy-Wouthuysen representation is determined. It agrees with the Lagrangian obtained by Pomeransky, Khriplovich, and Sen'kov. The validity of the Corben-Schwinger equations is confirmed. However, an attempt to generalize these equations in order to take into account the own quadrupole moment of particles was not successful. The known second-order wave equations are incorrect because they contain non-Hermitian terms. The correct second-order wave equation is derived.

1 Introduction

Investigations of interaction of spin-1 particles with an electromagnetic field are very important for the high energy physics. These investigations make possible to verify some spin-1 particle theories. There are many works where a consistency of spin-1 particle theories has been considered (see Ref. [1] and references therein). However, these works do not give us final conclusions. Terms bilinear in spin have been calculated by Pomeransky, Khriplovich, and Sen'kov for particles of any spin [2, 3]. In the present work, we verify some generalizations of the Proca equations. We transform the Hamilton operator to the block-diagonal form (diagonal in two spinors), which defines the Foldy-Wouthuysen (FW) representation [4]. This representation is very convenient for analysis of spin effects and for the semiclassical transition. The obtained result is compared with both classical [5] and Pomeransky-Khriplovich-Sen'kov (PKS) [2, 3] approaches.

2 Equations for spin-1 particles

For the first time, the equations for vector mesons have been found by Proca [6]. The wave function of the Proca equations has ten components. Corben and Schwinger [7] showed how to include an anomalous magnetic dipole term in the Proca equations, and Young and Bludman [8] took into account the own electric quadrupole moment.

Many first-order wave equations are equivalent [8, 9]. There exist also second-order wave equations.

Several components of the Proca equations can be expressed in terms of the others. As a result, the equations for the ten-component wave function can be reduced to the equation for the six-component one (the generalized Sakata-Taketani equation [8, 10]). As the components of the reduced wave function are two spinors, the wave function of the generalized Sakata-Taketani equation is a bispinor. This equation is very convenient for both a semiclassical transition and an investigation of spin dynamics.

Soon after the appearance of the Proca theory, the problem of its consistency was stated [11]. There are many works where several difficulties of spin-1 particle theories were investigated (e.g., complex energy modes for particles in a uniform magnetic field [1, 12, 13, 14]). In the above mentioned works the problem of consistency of spin-1 particle theories was solved qualitatively. However, there exists the more exact criterium of validity of any particle theory. As is shown in Refs. [2, 3, 15], the spin motion of particles with arbitrary spin is described by the Bargmann-Michel-Telegdi (BMT) equation [16]. The Lagrangian obtained in Refs. [2, 3] can be used for both finding the general equation of spin motion in nonuniform fields [17] and checking the relativistic wave equations.

3 Foldy-Wouthuysen transformation for spin-1 particles

The FW transformation for spin-1 particles has some peculiarities. The Hamiltonian for spin-1 particles is pseudo-Hermitian, that is, it satisfies the conditions:

$$\mathcal{H}=\rho_3\mathcal{H}^\dagger\rho_3, \quad \mathcal{H}^\dagger=\rho_3\mathcal{H}\rho_3.$$

The wave function is a six-component bispinor. The operator U, transforming the wave function to any other representation, should be pseudo-unitary:

$$U^{-1}=\rho_3U^\dagger\rho_3, \quad U^\dagger=\rho_3U^{-1}\rho_3.$$

The initial Hamiltonian is determined by the generalized Sakata-Taketani equation which can be written in the form

$$\mathcal{H}=\rho_3\mathcal{M}+\mathcal{E}+\mathcal{O}, \quad \rho_3\mathcal{E}=\mathcal{E}\rho_3, \quad \rho_3\mathcal{O}=-\mathcal{O}\rho_3, \tag{1}$$

where $\mathcal{E}$ and $\mathcal{O}$ are the even and odd operators, commuting and anticommuting with ρ_3, respectively.

In the general case, the external field is not stationary and the operator $\mathcal{O}$ commutes neither with $\mathcal{M}$, nor with $\mathcal{E}$. In this case the operator $\mathcal{O}$ can be divided into two operators:

$$\mathcal{O}=\mathcal{O}_1+\mathcal{O}_2. \tag{2}$$

The operator $\mathcal{O}_1$ should commute with $\mathcal{M}$, and the operator $\mathcal{O}_2$ should be equal to zero for the free particle. Therefore, the operator $\mathcal{O}_2$ should be relatively small.

First, it is necessary to perform the unitary transformation with the operator

$$U=\frac{\epsilon+\mathcal{M}+\rho_3\mathcal{O}_1}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}}, \quad U^{-1}=\frac{\epsilon+\mathcal{M}-\rho_3\mathcal{O}_1}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}}, \quad \epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}_1^2}. \tag{3}$$

After this transformation, the Hamiltonian $\mathcal{H}'$ still contains odd terms proportional to the derivatives of the potentials. Let us write the operator $\mathcal{H}'$ as:

$$\mathcal{H}'=\rho_3\epsilon+\mathcal{E}'+\mathcal{O}', \quad \rho_3\mathcal{E}'=\mathcal{E}'\rho_3, \quad \rho_3\mathcal{O}'=-\mathcal{O}'\rho_3, \tag{4}$$

where ϵ is defined by Eq. (3). The odd terms are small compared to both ϵ and the initial Hamiltonian $\mathcal{H}$. This circumstance allows us to apply the usual scheme of the nonrelativistic FW transformation [4, 18, 19].

Second, the transformation should be performed with the following operator:

$$U' = \exp{(iS')}, \quad S' = -\frac{i}{4}\rho_3\left\{\mathcal{O}', \frac{1}{\epsilon}\right\}_+ = -\frac{i}{4}\left[\frac{\rho_3}{\epsilon}, \mathcal{O}'\right], \tag{5}$$

where $\{\ldots,\ldots\}_+$ is an anticommutator and $[\ldots,\ldots]$ is a commutator. The further calculations are similar to those performed for spin-1/2 particles [4, 18, 19]. If only major corrections are taken into account, then the transformed Hamiltonian equals

$$\mathcal{H}'' = \rho_3\epsilon + \mathcal{E}' + \frac{1}{4}\rho_3\left\{\frac{1}{\epsilon}, \mathcal{O}'^2\right\}_+. \tag{6}$$

This is the Hamiltonian in the FW representation. To obtain the desired accuracy, the calculation procedure with transformation operator (5) should be repeated multiply.

4 Hamiltonian for spin-1 particles in a nonuniform electromagnetic field

Young and Bludman [8] have included terms describing the own quadrupole moment of particles in the Corben-Schwinger equations (CS) [7] and have made the Sakata-Taketani transformation [10]. The generalized Sakata-Taketani equation obtained in Ref. [8] defines the Hamiltonian acting on the six-component bispinor. This equation is similar to the Dirac equation for spin-1/2 particles. Therefore, it is useful to perform the FW transformation. In this section, we perform this transformation in the case without the own quadrupole moment of particles.

The method described above is used for finding the transformed Hamiltonian to within first-order terms in the field potentials (Φ and $\mathbf{A}$), strengths ($\mathbf{E}$ and $\mathbf{H}$), and first-order derivatives of the electric field strength. The terms of the second order and higher in the field potentials, strengths and their derivatives and the first-order terms with derivatives of all orders of the magnetic field strength and with derivatives of the second order and higher of the electric field strength will be omitted. The external field is considered to be stationary.

In this approximation, the basic generalized Sakata-Taketani equation for the Hamiltonian takes the form [8]

$$\begin{gathered}\mathcal{H} = e\Phi + \rho_3 m + i\rho_2\frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - (\rho_3 + i\rho_2)\frac{1}{2m}(\mathbf{D}^2 + e\mathbf{S}\cdot\mathbf{H}) - \\ (\rho_3 - i\rho_2)\frac{e\kappa}{2m}(\mathbf{S}\cdot\mathbf{H}) - \frac{e\kappa}{2m^2}(1+\rho_1)\Big[(\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) - i\mathbf{S}\cdot[\mathbf{E}\times\mathbf{D}] - \mathbf{E}\cdot\mathbf{D}\Big] + \\ \frac{e\kappa}{2m^2}(1-\rho_1)\Big[(\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot[\mathbf{D}\times\mathbf{E}] - \mathbf{D}\cdot\mathbf{E}\Big],\end{gathered} \tag{7}$$

where κ =const and $\mathbf{D} = \nabla - ie\mathbf{A}$.

We can introduce the g factor to describe the anomalous magnetic moment (AMM). In this case, $g=\kappa+1$. The Hamiltonian in the FW representation has the form

$$\begin{gathered}
\mathcal{H}''=\rho_3\epsilon'+e\Phi+\frac{e}{4m}\Biggl[\Biggl\{\left(\frac{g-2}{2}+\frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'},(\mathbf{S}\cdot[\boldsymbol{\pi}\times\mathbf{E}]-\mathbf{S}\cdot[\mathbf{E}\times\boldsymbol{\pi}])\Biggr\}_+-\\
\rho_3\left\{\left(g-2+\frac{2m}{\epsilon'}\right),\mathbf{S}\cdot\mathbf{H}\right\}_++\rho_3\left\{\frac{g-2}{2\epsilon'(\epsilon'+m)},\{\mathbf{S}\cdot\boldsymbol{\pi},\boldsymbol{\pi}\cdot\mathbf{H}\}_+\right\}_+\Biggr]+\\
\frac{e(g-1)}{4m^2}\left\{\left(\mathbf{S}\cdot\nabla-\frac{1}{\epsilon'(\epsilon'+m)}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\nabla)\right),\left(\mathbf{S}\cdot\mathbf{E}-\frac{1}{\epsilon'(\epsilon'+m)}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{E})\right)\right\}_++\\
\frac{e}{8m^2}\left\{\frac{1}{\epsilon'(\epsilon'+m)}\left(g-1+\frac{m}{\epsilon'+m}\right),\Bigl\{\mathbf{S}\cdot[\boldsymbol{\pi}\times\nabla],\mathbf{S}\cdot[\boldsymbol{\pi}\times\mathbf{E}]\Bigr\}_+\right\}_+-\\
\frac{e(g-1)}{2m^2}\nabla\cdot\mathbf{E}+\frac{e}{4m^2}\left\{\frac{1}{\epsilon'^2}\left(g-1+\frac{m^2}{4\epsilon'^2}\right),(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\mathbf{E})\right\}_+,\qquad \epsilon'=\sqrt{m^2+\boldsymbol{\pi}^2},
\end{gathered}\tag{8}$$

where $\boldsymbol{\pi}=-i\mathbf{D}=-i\nabla-e\mathbf{A}$ is the kinetic momentum operator.

The g factor of $g=g_{Pr}=1$ corresponds to the Proca particle. Nevertheless, the normal g factor is equal to 2 [2, 3].

For the stationary electric field, the operators $\mathbf{S}\cdot\nabla$ and $\mathbf{S}\cdot\mathbf{E}$ commute, because $\mathbf{E}=-\nabla\Phi$.

The transition to the semiclassical approximation consists of averaging the Hamilton operator over the wave functions of stationary states [17]. In the semiclassical approximation, the Hamiltonian is expressed by the relation

$$\begin{gathered}
\mathcal{H}''=\rho_3\epsilon'+e\Phi+\frac{e}{2m}\Biggl[\left(g-2+\frac{2}{\gamma+1}\right)\Bigl(\mathbf{S}\cdot[\mathbf{v}\times\mathbf{E}]\Bigr)-\\
\left(g-2+\frac{2}{\gamma}\right)\mathbf{S}\cdot\mathbf{H}+\frac{(g-2)\gamma}{\gamma+1}(\mathbf{S}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{H}\}\Biggr]+\\
\frac{e(g-1)}{2m^2}\left[\mathbf{S}\cdot\nabla-\frac{\gamma}{\gamma+1}(\mathbf{S}\cdot\mathbf{v})(\mathbf{v}\cdot\nabla)\right]\left[\mathbf{S}\cdot\mathbf{E}-\frac{\gamma}{\gamma+1}(\mathbf{S}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{E})\right]+\\
\frac{e\gamma}{2m^2(\gamma+1)}\left(g-1+\frac{1}{\gamma+1}\right)\Bigl(\mathbf{S}\cdot[\mathbf{v}\times\nabla]\Bigr)\Bigl(\mathbf{S}\cdot[\mathbf{v}\times\mathbf{E}]\Bigr)-\\
\frac{e(g-1)}{2m^2}\nabla\cdot\mathbf{E}+\frac{e}{2m^2}\left(g-1+\frac{1}{4\gamma^2}\right)(\mathbf{v}\cdot\nabla)(\mathbf{v}\cdot\mathbf{E}),
\end{gathered}\tag{9}$$

where $\gamma=\epsilon'/m$ is the Lorentz factor and $\mathbf{v}=\boldsymbol{\pi}/\epsilon'$ is the velocity. This relation is in the best compliance with the formulas for the Lagrangian of particles of arbitrary spin calculated in Refs. [2, 3]. Formulas (8),(9) contain spin-independent terms proportional to the derivatives of $\mathbf{E}$. These terms have not been calculated in [2, 3].

The perfect agreement between Hamiltonian (9) and the Lagrangian derived in Refs. [2, 3] bring the perfect agreement between the corresponding equations of spin motion.[1)] However, these equations disagree with the well-known Good-Nyborg equation [20, 21]. The part of Hamiltonian (9) determining the quadrupole interaction is in accord with the classical Hamiltonian derived in Ref. [5] for relativistic particles of any spin in an electromagnetic field.

[1)] The spin motion equation corresponding to this Lagrangian was obtained in Ref. [17].

Thus, the FW Hamiltonian determined on the basis of the CS equations is fully consistent with both the PKS theory [2, 3] and the classical Hamiltonian [5]. Therefore, the Proca and CS equations correctly describe, at least, weak-field effects.

5 Own quadrupole moment of particles

Spin-1 particles can possess the own quadrupole moment. The corresponding terms added to the Lagrangian should be bilinear in the meson field variables U_μ and $U_{\mu\nu}$, and linear in the derivatives of the electromagnetic field $\partial_\lambda F_{\mu\nu}$ [8]. The choice of these terms is strongly restricted by the Maxwell equations. As a result, there exists the only form of the additional terms describing the own quadrupole moment of particles [8].

Generalized Sakata-Taketani Hamiltonian (7) can be supplemented with the terms

$$\Delta\mathcal{H} = \frac{eq}{4m^2}\left[(S_iS_j + S_jS_i)\frac{\partial E_i}{\partial x_j} - 2\frac{\partial E_i}{\partial x_i}\right] \equiv \frac{eq}{4m^2}\left[\{(\mathbf{S}\cdot\nabla),(\mathbf{S}\cdot\mathbf{E})\}_+ - 2\nabla\cdot\mathbf{E}\right], \quad (10)$$

where q=const. Operator (3) defining the unitary transformation remains unchanged. The corresponding terms added to the FW Hamiltonian are given by

$$\begin{gathered}\Delta\mathcal{H}'' = -\frac{Q}{2}\Biggl[(\mathbf{S}\cdot\nabla)(\mathbf{S}\cdot\mathbf{E}) - \frac{1}{\epsilon' m(\epsilon'+m)^2}(\mathbf{S}\cdot\boldsymbol{\pi})^2(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\mathbf{E}) + \\ \frac{\epsilon'-m}{4\epsilon' m(\epsilon'+m)}\left(\Bigl\{\mathbf{S}\cdot\boldsymbol{\pi},(\boldsymbol{\pi}\cdot\nabla)(\mathbf{S}\cdot\mathbf{E})\Bigr\}_+ + \Bigl\{\mathbf{S}\cdot\boldsymbol{\pi},(\mathbf{S}\cdot\nabla)(\boldsymbol{\pi}\cdot\mathbf{E})\Bigr\}_+\right) - \nabla\cdot\mathbf{E}\Biggr],\end{gathered} \quad (11)$$

where $Q = -eq/m^2$. Eq. (11) disagrees with both Eq. (8) and the relativistic expression for the Lagrangian obtained in Refs. [2, 3].

The classical description of the quadrupole interaction of relativistic particles was given in Ref. [5]. The results obtained in this work are in agreement with Eqs. (8),(9) and contradict Eq. (11).

6 Relativistic wave equations of the second order

The usual way of derivation of second-order relativistic wave equations consists in an elimination of some components of the wave function. This approach results in appearance of non-Hermitian terms [17]. The presence of such terms can lead to both complex values of the particle energy and a nonorthogonality of the wave functions. Therefore, correct wave equations should be Hermitian [17, 22].

To obtain the correct second-order wave equation, the method elaborated in Ref. [23] can be used. In Ref. [23] the connection between first-order and second-order wave equations was found. The first-order wave equation can be written in the form

$$\mathcal{H} = \rho_3\epsilon' + W. \quad (12)$$

As follows from the results obtained in [23], the approximate form of the corresponding second-order wave equation is given by

$$\left[\left(i\frac{\partial}{\partial t} - W - \frac{1}{16}\left\{\frac{1}{\epsilon'^4},(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\nabla)W\right\}_+\right)^2 - \boldsymbol{\pi}^2 - m^2\right]\psi = 0. \quad (13)$$

The use of Eqs. (12),(13) makes possible to find the second-order wave equation for relativistic spin-1 particles interacting with an electromagnetic field. Such an equation corresponding to the first-order equation (8) is Hermitian and has the form

$$
\begin{gathered}
\left[\left(i\frac{\partial}{\partial t}-V\right)^2-\boldsymbol{\pi}^2-m^2\right]\psi=0,\\
V=e\Phi+\frac{e}{4m}\left[\left\{\left(\frac{g-2}{2}+\frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'},(\mathbf{S}\cdot[\boldsymbol{\pi}\times\mathbf{E}]-\mathbf{S}\cdot[\mathbf{E}\times\boldsymbol{\pi}])\right\}_+-\right.\\
\left.\rho_3\left\{\left(g-2+\frac{2m}{\epsilon'}\right),\mathbf{S}\cdot\mathbf{H}\right\}_++\rho_3\left\{\frac{g-2}{2\epsilon'(\epsilon'+m)},\{\mathbf{S}\cdot\boldsymbol{\pi},\boldsymbol{\pi}\cdot\mathbf{H}\}_+\right\}_+\right]+\\
\frac{e(g-1)}{2m^2}\Bigl(\mathbf{S}\cdot\nabla-\frac{1}{\epsilon'(\epsilon'+m)}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\nabla)\Bigr)\Bigl(\mathbf{S}\cdot\mathbf{E}-\frac{1}{\epsilon'(\epsilon'+m)}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{E})\Bigr)+\\
\frac{e}{4m^2}\left\{\frac{1}{\epsilon'(\epsilon'+m)}\left(g-1+\frac{m}{\epsilon'+m}\right),\Bigl(\mathbf{S}\cdot[\boldsymbol{\pi}\times\nabla]\Bigr)\Bigl(\mathbf{S}\cdot[\boldsymbol{\pi}\times\mathbf{E}]\Bigr)\right\}_+-\\
\frac{e(g-1)}{2m^2}\nabla\cdot\mathbf{E}+\frac{e(g-1)}{4m^2}\left\{\frac{1}{\epsilon'^2},(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\mathbf{E})\right\}_+.
\end{gathered}
\tag{14}
$$

Unfortunately, it is difficult to obtain a compact four-dimensional form of Eq. (14).

7 Discussion and summary

Thus, we have theoretically verified the relativistic wave equations for spin-1 particles in nonuniform electric and uniform magnetic fields. The Hamilton operator in the FW representation is determined. In contrast to [2, 3], we also took into account the spin-independent terms proportional to $\partial E_i/\partial x_j$, which allow description of the contact interaction. Our analysis shows that the first-order Proca and CS equations correctly describe, at least, weak-field effects.

The CS equations can be derived with the first-order Lagrangian of spin-1 particles in an electromagnetic field [8]. However, the attempt to introduce the own quadrupole moment by adding appropriate second-order terms to the Lagrangian [8] does not lead to the correct result. The PKS approach makes possible to find the right Lagrangian for particles of any spin having the own quadrupole moment. The validity of this Lagrangian is confirmed by the comparison with the classical description given in [5]. This conclusion poses a serious problem.

The Good-Nyborg equation [20, 21] incorrectly describes the spin motion.

The known second-order wave equations are incorrect, because they contain non-Hermitian terms. This would result in complex values of the particle energy and in the nonorthogonality of the wave functions [17]. The correct second-order wave equation is derived by the method elaborated in [23].

Acknowledgements

This work was supported by the grant of the Belarusian Republican Foundation for Fundamental Research No. Φ03-242.

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