On boundary conditions as Dirac constraints

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Abstract

The idea to apply to boundary conditions Dirac methods developed for constrains is studied.

Unfortunately, almost this title has been already used [1]. To our knowledge the idea to treat boundary conditions as constraints or as gauge conditions was mentioned first in paper [2]. It was proposed there to improve the standard formula for the Poisson brackets by some boundary terms in order to remove a restriction of applying these brackets to nontrivial boundary problems. This restriction follows from the fact that standard Poisson brackets fulfill the Jacobi identity only modulo divergences

$$\{\{F,G\},H\}+\{\{H,F\},G\}+\{\{G,H\},F\}=\int \mathrm{Div}$$

Some other optimistic hopes on future progress in this field together with the proposal on the opportunity to work with boundary conditions as with constraints were also expressed in article [2].

It is difficult to reject the fact that much time have passed since the appearance of the extremely popular textbook [3] and that it is not always possible to avoid boundary terms simply by moving boundaries of any physical system to infinity as recommended there. One of these problems is dynamics of the open string. In particular, it was observed that in presence of antisymmetric tensor field the string ending on a D-brane is described by the standard equations, but with new boundary conditions which are partially Dirichlet and partially mixed. The Hamiltonian approach to this problem has led some authors to an idea that the noncommutative geometry of D-branes could be derived from it.

How this could be possible? Let us take the standard action for open relativistic string (with $2\pi\alpha'=1$)

$$S_0=\int\limits_{t_1}^{t_2}\int\limits_0^{\pi}\sqrt{|\det G_{AB}|}d\tau d\sigma,\quad G_{AB}=g_{\mu\nu}X^{\mu}_{,A}X^{\nu}_{,B}.$$

In the conformal gauge this action is as follows

$$S_0=\frac{1}{2}\int\limits_{t_1}^{t_2}d\tau\int\limits_0^{\pi}\left(\dot X^2-X'^2\right)d\sigma.$$

The relevant Hamiltonian variables X^{μ}, $p_{\mu}(=\dot X_{\mu})$ are canonically conjugate to each other and the Hamiltonian takes the following form

$$H_0=\int\limits_0^{\pi}\left(\frac{p^2}{2}+\frac{X'^2}{2}\right)d\sigma,$$

and generate the following equations

$$\dot{X}^{\mu} = \{X^{\mu}, H\} = p^{\mu}, \quad \dot{p}_{\mu} = \{p_{\mu}, H\} = X_{\mu}''.$$

Let us consider a flat (for simplicity) D-brane fixed by equations $x^a = 0$. Let the ends of a string are forced to live on it, but they are free in other directions. This leads us to boundary conditions of the following form

$$(X^i)' = 0, \quad i = 0, 1, \dots, p-1,$$

$$X^a = 0, \quad a = p, \dots, d-1.$$

Let also there be a constant (for simplicity) antisymmetric field $B_{\mu\nu}$, and there be abelian gauge field A^i interacting with sting boundaries

$$S_1 = \left[\int\limits_{t_1}^{t_2} A_i \dot{X}^i d\tau \right]_0^{\pi}.$$

Then we can write these interaction terms in the following form

$$S_1 = \frac{1}{2} \int\limits_{t_1}^{t_2} d\tau \int\limits_0^{\pi} F_{ij} \dot{X}^i X'^j d\sigma,$$

and the total action is given as follows

$$S = \frac{1}{2} \int\limits_{t_1}^{t_2} d\tau \int\limits_0^{\pi} \left(\dot{X}^2 - X'^2 + \epsilon^{AB} \mathcal{F}_{\mu\nu} \partial_A X^{\mu} \partial_B X^{\nu} \right) d\sigma.$$

where

$$\mathcal{F}_{\mu\nu} = B_{\mu\nu} - F_{\mu\nu}.$$

The canonical variables for the new action have changed. The momenta conjugate to string coordinates are as follows

$$P_{\mu} = p_{\mu} + \mathcal{F}_{\mu\nu} X'^{\nu}.$$

This transformation of momenta has the special form

$$P_A = p_A + \frac{\delta F[q]}{\delta q^A},$$

and so belongs to the same class as the celebrated Ashtekar transformation in General Relativity

$$A_i^a = iK_i^a + \Gamma_i^a.$$

As it was shown by us before [4], such a transformation is strictly canonical in mechanics, but not in field theory, because the Euler-Lagrange variational derivatives commute only up to divergences.

Therefore, the old momenta do not commute for the new action

$$\{p_i(\sigma), p_j(\sigma')\} = \mathcal{F}_{ij}\theta'\delta(\sigma,\sigma'),$$

where

$$\theta(\sigma) = \begin{cases} 1 & \text{if } \sigma \in [0,\pi]; \\ 0 & \text{otherwise}, \end{cases}$$

which corresponds to appearance of boundary terms

$$\Omega = \int\limits_0^\pi \left(\delta X^\mu \wedge \delta p_\mu + (\mathcal{F}_{ij}\delta X^i \wedge \delta X^j)'\right) d\sigma$$

in the symplectic form.

In principle it is possible to construct the string Hamiltonian formalism for this problem in two equivalent ways: with the new or with the old momenta. In the first case the momenta are commuting, boundary terms in the Poisson brackets for the main variables do not appear and boundary conditions are simultaneously both "differentiability" conditions for the Hamiltonian — the Regge-Teitelboim approach does work [5]. In the second case more complicated scheme is realized which mixes surface terms of the Hamiltonian variation and of the Poisson brackets (or the Poisson bivector), boundary conditions then appear as the regularity condition for the Hamiltonian vector field [6]. In both cases the practical realization of the idea of boundary conditions as constraints (or gauges) and of applying to them the Dirac method (Dirac brackets) runs into difficulties.

One of them is the fact that the boundary condition does not look like an equality for some local functional whereas Poisson brackets are defined only for such objects. Nevertheless, gauges or constraints may be integrated with some test function and become local functionals. But for boundary we do not have suitable test functions. In one-dimensional space (for example, a string) it is nevertheless possible to represent a boundary condition as an equation for some local functional. We can introduce test function $\xi(\sigma)$ which is equal to unity at one end of the string and equal to zero at another which also have all derivatives equal to zero at both ends. Then the corresponding functional is an integral of the spatial derivative of a product of such a test function and the boundary condition.

Let us consider this in detail. First we remind the modified formula for the Poisson brackets (in the special case of one-dimensional space))

$$\{F, G\} = \int D_{i+j}\left(E^i_{q^A}(f)E^j_{p_A}(g) - E^i_{p^A}(f)E^j_{q_A}(g)\right) dx,$$

here local functionals F, G are given as follows

$$F = \int f\left(q^A(x), \frac{dq^A}{dx}, \ldots, p_A(x), \frac{dp_A}{dx}, \ldots\right) dx,$$

and higher Eulerian operators are

$$E^i_{q^A} = \sum_{k=i}^{\infty}(-1)^{i+k}\binom{k}{i} D_{k-i}\frac{\partial}{\partial q^{A(k)}}.$$

Let us start with the standard string Hamiltonian and with the standard canonical structure. Then we put boundary conditions of the following form

$$\Phi_1 \equiv X^a = 0, \quad a = p, \ldots, d-1,$$

$$\Phi_2 \equiv X'^i + F^i{}_j p^j = 0, \quad i = 0, \ldots, p-1,$$

at both string ends as constraints (or gauges), i.e. add them to Hamiltonian H_0 with some Lagrange multipliers. Next we calculate the Poisson brackets of the constraints with new Hamiltonian $H = H_0 + v_1\Phi_1 + v_2\Phi_2$:

$$\{\Phi_1, H\} \equiv \{\int_0^\pi (\xi X^a)' d\sigma, H\} = p^a,$$

$$\{\Phi_2, H\} \equiv \{\int_0^\pi (\xi(X'^i + F^i{}_j p^j))' d\sigma, H\} = (p^i)'$$

This gives us new boundary conditions

$$\Phi_3 \equiv p_a = 0, \quad \Phi_4 \equiv (p_i)' = 0,$$

which could also be represented as conditions on some local functionals

$$\int_0^\pi (\xi p_a)' d\sigma = 0, \quad \int_0^\pi (\xi p_i)'' d\sigma = 0.$$

The Poisson brackets of boundary conditions with each other are zero, so the constraints occur as of the first class in this approach.

Repeating this test for self-consistence of the constraints (i.e. boundary conditions) with the Hamiltonian we do not obtain any new equations

$$\{\Phi_3, H\} = 0, \quad \{\Phi_4, H\} = 0.$$

This means that there are first class constraints in the Hamiltonian, but they do not modify the Hamiltonian equations. This situation is not possible neither in mechanics nor in field theory where constraints take place in some physical volume and not on the boundary. Of course, no Dirac brackets arise here.

Another opportunity is a transition from field theory to mechanics by discretization, i.e. by replacing an infinite number of degrees of freedom by a finite number of them [7]. Here unfortunately we get some ambiguity. The discretized action may include terms which tend to zero in the continuity limit. Nevertheless these terms may be necessary for getting the "right" answer.

The third approach [8] is inspired by holography. It is proposed to consider a boundary condition first as an ordinary gauge i.e. as a condition to be valid at any internal point of the physical domain. Afterward Dirac brackets are calculated and all boundary contributions are discarded in them. The rest are declared to be Dirac brackets on the boundary whereas in the bulk Poisson brackets remain the same. We are not quite sure that this approach is correct, though the authors claim that it reproduces all the known results and predicts new ones.

The forth method [9] suggests a simultaneous fixing of surface terms in the Poisson brackets and in the Hamiltonian. In particular, it is claimed that the noncommutativity of D-brane coordinates could be avoided even for arbitrary (i.e. non-constant) B-field. The explicit form of action is not used at all, the method is purely Hamiltonian. The authors observe some ambiguity in solving the problem. They claim

1. exact Jacobi identity for the Poisson brackets;

2. zero Poisson brackets for boundary conditions with any canonical variable.

The brackets here are distributions and not local functionals. Boundary conditions become functionals after their integration with the boundary δ-function.

We are still unable to say that the method of treatment boundary conditions as the Dirac constraints lets us solve unambiguously the problem of finding boundary terms in the Poisson brackets for an open string interacting with D-brane in presence of an antisymmetric tensor field. Maybe some more realistic treatment of D-brane is required which accounts for the D-brane dynamics.

References

[1] M.M. Sheikh-Jabbari and A. Shirzad, Eur.Phys.J. C19 (2001) 383

[2] V.O. Soloviev, J. Math. Phys. 34 (1993) 5747

[3] L. Landau and E. Lifshitz, Field Theory

[4] V.O. Soloviev, Phys. Lett. B292 (1992) 30

[5] T. Regge and C. Teitelboim, Ann. Phys. 88 (1974) 286

[6] V.O. Soloviev, Phys. Rev. D55 (1997) 7793

[7] Chong-Sun Chu and Pei-Ming Ho, Nucl.Phys. B550 (1999) 151

[8] J.M. Romero and J.D. Vergara, Boundary conditions as constraints, hep-th/0212035

[9] Liu Zhao and Wenli He, On open string in generic background, hep-th/0306203