The physical degrees of freedom of the RS2 model

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Abstract

Equations for linearized gravity in the RS2 model with matter on the brane are solved exactly in a convenient and physically transparent gauge. It is shown that due to the physical boundary conditions at infinity for metric fluctuations and gauge functions the radion field cannot be completely gauged out in the case of presence of matter on the brane.

1 Introduction

The Randall-Sundrum model with one brane (RS2 model) [1] represents a possible way of localizing gravity on a brane in the case of one infinite extra dimension. The model describes gravity in a five-dimensional space-time E with one brane embedded into it. Let us denote the coordinates in E by $\{x^M\} \equiv \{x^\mu, x^4\}$, $M = 0,1,2,3,4$, $\mu = 0,1,2,3$, the coordinate $x^4 \equiv y$ parameterizing the fifth infinite dimension. The brane is located at $y = 0$, and all the fields possess a symmetry under the reflection $y \leftrightarrow -y$, which reads

$$g_{\mu\nu}(x,-y) = g_{\mu\nu}(x,y), \quad g_{\mu 4}(x,-y) = -g_{\mu 4}(x,y), \quad g_{44}(x,-y) = g_{44}(x,y). \tag{1}$$

It is evident, that the RS2 model can be obtained from the Randall-Sundrum model with two branes (RS1 model) [2] by pushing the negative tension brane to infinity.

It is universally recognized that the scalar mode of the gravitational field - the radion field, which plays an important role in the RS1 model (it describes the oscillations of the branes with respect to each other) is absent in the RS2 model, since there is only one brane in this model. In papers [1, 3, 4] it was noted that the 44-component of the metric fluctuations, which corresponds to this scalar mode, could be gauged out. However, in this gauge the brane is located not at $y = 0$, but at $y = \xi(x)$. Obviously, this formulation, which was called the "bent-brane" formulation, destroys the reflection symmetry (1), which makes the approach based on this gauge inconsistent. This fact was noted in [5]. In paper [6] it was observed that gauging out the radion field in the straight brane formulation with matter on the brane leads to unphysical solutions, which diverge at infinity. Thus, it looks as if this field were of particular importance in the RS2 model. In the present paper we are going to study the role of the radion in the RS2 model more thoroughly. In the beginning let us briefly discuss the main features of the RS2 model.

The action of the model is

$$S = \frac{1}{16\pi\hat{G}} \int_E (R - \Lambda)\sqrt{-g}\, d^4x dy + V \int_E \sqrt{-\tilde{g}}\delta(y) d^4x dy, \tag{2}$$

where $\tilde{g}_{\mu\nu}$ is the induced metric on the brane and V is the brane tension. The signature of the metric g_{MN} is chosen to be $(-,+,+,+,+)$.

The Randall-Sundrum solution for the metric is given by

$$ds^2 = g_{MN}dx^M dx^N = e^{2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{3}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and the function $\sigma(y) = -k|y|$. The parameter k is positive and has the dimension of mass; the parameters Λ and V are related to it as follows:

$$\Lambda = -12k^2, \quad V = -\frac{3k}{4\pi\hat{G}}.$$

Thus, the brane has a positive energy density.

We parameterize the metric g_{MN} as

$$g_{MN} = \gamma_{MN} + \hat{\kappa}h_{MN}, \tag{4}$$

where $\hat{G}$ is the five-dimensional gravitational constant, $\hat{\kappa} = \sqrt{16\pi\hat{G}}$ and h_{MN} are the metric fluctuations. Substituting this parameterization into (2) and retaining the terms of the zeroth order in $\hat{\kappa}$, we get the second variation action of this model. In [7] the second variation action for the RS1 model was obtained. We can easily apply this result to the RS2 model just by changing the definition of $\sigma(y)$ and of its derivatives.

The action is invariant under the gauge transformations

$$\begin{aligned}
h'_{\mu\nu}(x) &= h_{\mu\nu}(x) - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu + 2\gamma_{\mu\nu}\partial_4\sigma\xi_4), && (5)\\
h'_{\mu4}(x) &= h_{\mu4}(x) - (\partial_\mu\xi_4 + \partial_4\xi_\mu - 2\partial_4\sigma\xi_\mu), && (6)\\
h'_{44}(x) &= h_{44}(x) - 2\partial_4\xi_4, && (7)
\end{aligned}$$

where the functions $\xi^M(x,y)$ satisfy the symmetry conditions

$$\xi^\mu(x,-y) = \xi^\mu(x,y), \quad \xi^4(x,-y) = -\xi^4(x,y). \tag{8}$$

2 Equations of motion and gauge conditions

The interaction of linearized gravity with matter on the brane has the standard form

$$\frac{\hat{\kappa}}{2}\int_{brane} h^{\mu\nu}(x,0)t_{\mu\nu}(x)dx, \tag{9}$$

$t_{\mu\nu}(x)$ denoting the energy-momentum tensor of the matter. The corresponding equations of motion look as follows:

1) $\mu\nu$-component

$$\begin{aligned}
&\frac{1}{2}(\partial_\rho\partial^\rho h_{\mu\nu} - \partial_\mu\partial^\rho h_{\rho\nu} - \partial_\nu\partial^\rho h_{\rho\mu} + \partial_4\partial_4 h_{\mu\nu}) - && (10)\\
-\;& 2k^2 h_{\mu\nu} + \frac{1}{2}\partial_\mu\partial_\nu\tilde{h} + \frac{1}{2}\partial_\mu\partial_\nu h_{44} - \partial_4\sigma(\partial_\mu h_{\nu4} + \partial_\nu h_{\mu4}) - \frac{1}{2}\partial_4(\partial_\mu h_{\nu4} + \partial_\nu h_{\mu4}) + \\
+\;& \frac{1}{2}\gamma_{\mu\nu}\Big(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial_\rho\partial^\rho\tilde{h} - \partial_4\partial_4\tilde{h} - 4\partial_4\sigma\partial_4\tilde{h} - \partial_\rho\partial^\rho h_{44} + 12k^2 h_{44} + 3\partial_4\sigma\partial_4 h_{44} + \\
+\;& 2\partial^\rho\partial_4 h_{\rho4} + 4\,\partial_4\sigma\partial^\rho h_{\rho4}\Big) + (2kh_{\mu\nu} - 3k\gamma_{\mu\nu}h_{44})\,\delta(y) = -\frac{\hat{\kappa}}{2}t_{\mu\nu}(x)\delta(y),
\end{aligned}$$

2) $\mu4$-component

$$\partial_4(\partial_\mu \tilde{h} - \partial^\nu h_{\mu\nu}) - 3\partial_4\sigma\partial_\mu h_{44} - \partial^\nu\left(\partial_\mu h_{\nu 4} - \partial_\nu h_{\mu 4}\right) = 0, \tag{11}$$

3) 44-component

$$\frac{1}{2}(\partial^\mu\partial^\nu h_{\mu\nu} - \partial_\mu\partial^\mu\tilde{h}) - \frac{3}{2}\partial_4\sigma\partial_4\tilde{h} + 6k^2 h_{44} + 3\partial_4\sigma\partial^\mu h_{\mu 4} = 0, \tag{12}$$

where $\tilde{h} = \gamma^{\mu\nu}h_{\mu\nu}$ and all indices are raised with metric $\gamma^{\mu\nu}$.

In what follows, we will also use an auxiliary equation, which is obtained by multiplying the equation for 44-component by 2 and subtracting it from the contracted equation for $\mu\nu$-component. This equation contains $\tilde{h}$, $h_{\mu 4}$ and h_{44} only and has the form:

$$\partial_4\left(e^{2\sigma}\partial_4\tilde{h}\right) - 4\partial_4\left(e^{2\sigma}\partial_4\sigma h_{44}\right) - 2\partial_4(e^{2\sigma}\partial^\mu h_{\mu 4}) + \Box h_{44} = \frac{\hat{\kappa}}{3}\, t^\mu_\mu(x)\delta(y), \tag{13}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

First, we would like to emphasize that, in general, all fluctuations of metric must satisfy the physical boundary conditions at $y \to \pm\infty$, $x^i \to \pm\infty \quad (i = 1, 2, 3)$, i.e. they must be finite at spatial infinity. Any solution of equations (10)-(13) can be expressed as a sum of solutions of the corresponding inhomogeneous and homogeneous equations. Solutions of the homogeneous equaitons, which are of plane wave type, can be finite at spatial infinity, whereas solution of the inhomogeneous equations must vanish at infinity for physical reasons. These assumptions are reasonable - for example, the h_{00}-component is associated with Newton's potential, which must vanish at infinity (for the matter, which is localized in some finite domain). Below we will show that the fields of certain exact solutions (for example, with point-like matter sources) do satisfy these boundary conditions.

Obviously, the gauge functions $\xi^M(x, y)$ must be finite everywhere in E. It follows from the definition of the gauge transformations. Thus

$$\xi_\mu = e^{2\sigma}\eta_{\mu\nu}\xi^\nu|_{y\to\pm\infty} \to 0$$

and $\xi_4 = g_{44}\xi^4 = \xi^4$ is finite at $y \to \pm\infty$.

Now let us consider equation (13). By integrating it over y in the limits $(-\infty, \infty)$ and using the symmetry and the physical boundary conditions for the fields $h_{\mu\nu}$, $h_{\mu 4}$ and h_{44}, we find that the function $\varphi(x)$, defined by

$$\int_{-\infty}^{\infty} h_{44}(y)\, dy = \varphi(x), \tag{14}$$

is not equal to zero and satisfies the equation

$$2e^{2\sigma}\partial_4\tilde{h}|_{y\to\infty} - 4e^{2\sigma}\partial^\mu h_{\mu 4}|_{y\to\infty} + \Box\varphi(x) = \frac{\hat{\kappa}}{3}\,\eta^{\mu\nu}t_{\mu\nu}(x) \equiv \frac{\hat{\kappa}}{3}\,t(x). \tag{15}$$

We represent h_{44} as

$$h_{44}(y) = h_{44}^{hom}(y) + h_{44}^{inhom}(y), \tag{16}$$

where $h_{44}^{inhom}(y)$ is the solution of equation (13) with zero boundary conditions and $h_{44}^{hom}(y)$ is a solution of the corresponding homogeneous equation, which can be finite at $y \to \pm\infty$. The fields $h_{\mu\nu}$ and $h_{\mu 4}$ can be represented in the same manner as well. Then we get

$$2e^{2\sigma}\partial_4 \tilde{h}^{hom}|_{y\to\infty} - 4e^{2\sigma}\partial^{\mu} h_{\mu 4}^{hom}|_{y\to\infty} + \Box\varphi^{hom}(x) = 0, \tag{17}$$

$$\Box\varphi^{inhom}(x) = \frac{\hat{\kappa}}{3}\, t(x). \tag{18}$$

There is a good reason to guess that since $\varphi^{hom}(x)$ satisfies equation (17) it is finite. Taking ξ_4 of the form

$$\xi_4(x,y) = \frac{1}{4}\int_{-y}^{y} h_{44}^{hom}(x,y')dy', \tag{19}$$

we can gauge out $h_{44}^{hom}(y)$. Note that ξ_4 does not need to vanish at $y \to \pm\infty$. Having gauged out $h_{44}^{hom}(y)$, we are still left with $h_{44}(y) = h_{44}^{inhom}(y)$. Now that we have gauged out the plane wave component of the fluctuations of metric, these fluctuations must vanish at spatial infinity ($y \to \pm\infty$, $x^i \to \pm\infty \quad (i = 1,2,3)$), i.e must satisfy zero boundary conditions. This issue was discussed above. Correspondingly, we have to reconsider the boundary conditions for the gauge functions. Let us consider equation (6). It follows from this equation that if ξ_4 does not depend on four-dimensional coordinates x, we can satisfy the new physical boundary condition for the field $h_{\mu 4}$ without requiring ξ_4 to vanish at $y \to \pm\infty$. But in this case the last term in the r.h.s. of (5) does not satisfy the new physical boundary condition for the field $h_{\mu\nu}$ at $x^i \to \pm\infty$. This means that ξ_4 must be x-dependent and vanish at $x^i \to \pm\infty$. At the same time the form of the second term in the r.h.s. of (6) shows that ξ_4 must vanish at $y \to \pm\infty$ to satisfy the zero boundary conditions for the field $h_{\mu 4}$. Thus, now we have the following boundary conditions for ξ_M:

$$\xi_M|_{x^i, y\to\pm\infty} \to 0. \tag{20}$$

Obviously, the admissible gauge transformations do not alter these zero boundary conditions for the metric fluctuations, and therefore equation (15) holds in any gauge. This is an equation for a four-dimensional scalar field, which coincides with the equation for the radion field in the RS1 model with matter on the positive tension brane [8].

Thus, we arrive at the corollary that the radion field cannot be totally gauged out in the RS2 model, because otherwise the equations of motion for linearized gravity become inconsistent. In other words, this means that the gauge functions ξ_4, corresponding to the gauge $h_{44} = 0$, do not satisfy the boundary conditions at $y \to \pm\infty$, which is a good check of the consistency of our approach. In fact, this was noted in [6]. It was shown there that the solutions for the linearized gravity in the absence of the radion are unphysical, i.e. they diverge at $y = \pm\infty$.

We will use the following form of ξ_4 to impose an appropriate gauge on the field h_{44}:

$$\xi_4(x,y) = \frac{1}{4}\int_{-y}^{y} h_{44}(x,y')dy' - \frac{1}{4C}\int_{-y}^{y} F(y')dy' \int_{-\infty}^{\infty} h_{44}(x,y')dy', \tag{21}$$

where $F|_{y\to\pm\infty} = 0$ and $C = \int_{-\infty}^{\infty} F(y)dy$. Note that ξ_4 satisfies the symmetry and the boundary conditions. With the help of (21) we can pass to the gauge, in which

$$h_{44}(x,y) = F(y)\phi(x), \tag{22}$$

where

$$\phi(x) = \frac{1}{C} \int_{-\infty}^{\infty} h_{44}(x, y) dy \tag{23}$$

and depends on x only. It turns out to be convenient to choose $F(y) = e^{2\sigma} = e^{-2k|y|}$. Obviously, the field h_{44} satisfies the symmetry and the physical boundary conditions in this gauge. Moreover, we have no residual gauge transformations with ξ_4. We also note that since $\xi_4(x, 0) = 0$, the brane remains straight in this gauge, i.e. we *do not* use the bent-brane formulation [3, 4], which destroys the structure of the model (see [5]).

We would like to note that the gauge choice of the type (22) with an arbitrary finite even function $F(y)$ can be used in the RS1 model as well. For example, a gauge with $h_{44}(x, y) \sim e^{2k|y|}\, \phi(x)$ was used in [9].

Now let us discuss the gauge condition for the field $A_\mu = h_{\mu 4}$. Let us take the gauge function $\xi_\mu(x, y)$ in the following form:

$$\xi_\mu(x, y) = e^{2\sigma} \int_{-\infty}^{y} e^{-2\sigma} A_\mu \left(x^\nu, y'\right)\, dy'. \tag{24}$$

Of course, this definition makes sense, if the field A_μ is such that the integral in (24) is well convergent to provide an acceptable ($\sim e^{2\sigma}$) decrease of ξ_μ. One can easily see that due to the symmetry $A_\mu(x, -y) = -A_\mu(x, y)$ (see (1)), $\xi_\mu(x, y)$ satisfies the symmetry condition (8). Moreover, it is easy to see that $\xi_\mu(x, y)|_{y\to\pm\infty} \to 0$, at least in the sense of the principal value of the integral in eq. (24) (again due to the symmetry of A_μ). Finally, it is not difficult to check that the gauge transformation with ξ_μ given by (24) gauges the field A_μ out.

We think that this formal argumentation can be used in favor of the possibility to make the A_μ-field vanish everywhere. Moreover, with a different motivation, an expression similar to (24) was assumed to be well defined in [5]. Anyway, in all the papers concerning the RS2 model it is universally recognized that the field A_μ can be gauged away (see, for example, [6]). Thus, we also adhere to this opinion. As we will see later, equations of motion can be solved exactly in the gauge $A_\mu = 0$ (see also [6]).

After this gauge fixing we are still left with residual gauge transformations of the form

$$\partial_4 \left(e^{-2\sigma} \xi_\mu\right) = 0. \tag{25}$$

Now we are ready to solve equations of motion in the gauge

$$h_{\mu 4}(x, y) = 0, \quad h_{44}(x, y) = e^{2\sigma} \phi(x). \tag{26}$$

The gauge is very simple and is more transparent from the physical point of view, than that in [5], where the equations of motion for linearized gravity were solved exactly as well.

3 Solution of the equations of motion

The substitution, which allows us to solve equations of motion in the gauge (26), has the form

$$h_{\mu\nu} = b_{\mu\nu} + \frac{1}{2}\, e^{2\sigma} \gamma_{\mu\nu} \phi - \frac{1}{2k^2}\, \sigma\, e^{2\sigma} \partial_\mu \partial_\nu \phi. \tag{27}$$

Note that if $b_{\mu\nu}|_{y\pm\to\infty} \to 0$, then $h_{\mu\nu}|_{y\pm\to\infty} \to 0$. Substituting (27) into (11), (12), (13), using the physical boundary conditions for the field $b_{\mu\nu}$, the symmetry conditions and the gauge transformations (25) we get (in a similar way as it was made in [8])

$$\Box\,\phi = \frac{\hat{\kappa}k}{3}t. \tag{28}$$

and

$$b = 0, \quad \partial^{\mu} b_{\mu\nu} = 0, \tag{29}$$

where $b = \eta^{\mu\nu} b_{\mu\nu}$ and indices are raised with flat Minkowski metric $\eta^{\mu\nu}$. The conditions (29) define the gauge, which is usually called the transverse-traceless gauge. Having imposed this gauge, we are still left with residual gauge transformations

$$\xi_{\mu} = e^{2\sigma}\epsilon_{\mu}(x), \quad \Box\epsilon_{\mu} = 0, \quad \partial^{\mu}\epsilon_{\mu} = 0, \tag{30}$$

which are important for determining the number of degrees of freedom of the massless mode of $b_{\mu\nu}$.

Substituting (27) into (10) and using (29) and (28), we get the well-known equation

$$\frac{1}{2}\left(e^{-2\sigma}\Box\, b_{\mu\nu} + \partial_4\partial_4 b_{\mu\nu}\right) - 2k^2 b_{\mu\nu} + 2k b_{\mu\nu}\delta(y) = -\frac{\hat{\kappa}}{2}\,\delta(y)\left[t_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_{\mu}\partial_{\nu}}{\Box}\right)t\right]. \tag{31}$$

This equation is identical to the one obtained by Garriga and Tanaka [3]. It was solved exactly, for example, in [5], and the solution for ordinary (not tachyonic) matter on the brane looks like

$$b_{\mu\nu}(x,y) = \frac{1}{(2\pi)^4}\int_{p^2>0} e^{-i\eta_{\mu\nu}p^{\mu}x^{\nu}}\tilde{b}_{\mu\nu}(p,y)d^4p, \tag{32}$$

where for $p^2 = -p_0^2 + \vec{p}^2 > 0$ (which includes the static case $p_0 = 0$)

$$\tilde{b}_{\mu\nu}(p,y) = \left[\tilde{t}_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}\right)\tilde{t}\right]\frac{\hat{\kappa}}{2\sqrt{p^2}}\frac{K_2\left(\frac{\sqrt{p^2}}{k}e^{k|y|}\right)}{K_1\left(\frac{\sqrt{p^2}}{k}\right)}. \tag{33}$$

One can easily see that with an appropriate energy-momentum tensor $t_{\mu\nu}(x)$ (for example, that of a static point-like source) fields $h_{\mu\nu}(x,y)$ and $h_{44}(x,y)$ do decay to zero at the spatial infinity. We would like to note that (33) coincides with the corresponding formula for the RS1 model, obtained in [8], in the limit $R \to \infty$.

Now let us examine gravity on the brane. The fluctuations of the metric on the brane have the following form

$$h_{\mu\nu}(x,0) = b_{\mu\nu}(x,0) + \frac{\hat{\kappa}k}{6}\,\eta_{\mu\nu}\Box^{-1}t. \tag{34}$$

Using (31), one can easily see that equation (34) coincides with the solutions for gravity on the brane, obtained in [3, 4, 5, 6], although we did not use the "bent-brane" formulation, which was used in [3, 4].

Now we are ready to discuss the number of the independent degrees of freedom in the RS2 model. As we have shown, we cannot completely gauge away the radion field in the presence of matter on the brane (see (15)). We can only gauge out that part of the field, which corresponds to the solution of the equations of motion without matter on the brane. Obviously, if there is no matter on the brane, we deal with equations for the free fields, possessing solutions of the plane wave type (which do not vanish at infinity). It means that in this case the radion field can be gauged out and is no more an independent degree of freedom of the RS2 model. But the radion field appears, if we place matter sources on the brane, and it allows one to solve consistently the equations of motion.

This situation is analogous to that in electrodynamics. It is a common knowledge that longitudinal photons do not appear in the asymptotic states (on the mass shell), whereas their contribution is important in the radiative corrections (off the mass shell). The radion field is very similar to longitudinal photons: it is absent in the asymptotic states, but it is absolutely necessary for consistently describing the interaction off the mass shell.

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