Spontaneous generation of an effective interaction in a renormalizable model of quantum field theory

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N.N. Bogolubov quasi-averages method [1, 2] is the most consistent and effective method of studying of a spontaneous symmetry breaking problems.

The phenomenon of spontaneous symmetry breaking is decisive for formulation of the electroweak theory. The introduction of elementary scalar fields [3] is well-known to be the essence of the standard way of the breaking. Their self-interaction leads to redefinition of the vacuum in the theory and to existence of the scalar Higgs particle. However proposals based on a dynamical breaking of the initial symmetry without elementary Higgs scalars are also considered. This leads to effective four-fermion interaction of heavy quarks either to be postulated (see e.g. review [4]) or to be dynamically arisen [5, 6]. As a result the initial theory, which contains scale-invariant gauge interactions, transforms into a theory, which contains interactions with dimensional coupling constant as well, that explicitly breaks the scale invariance. In works being mentioned [5, 6] assumption was made on a possibility of existence of solutions of corresponding compensation equations. However this assumption was not duly justified. The purpose of the present work is to consider a simpler model, which would allow to have exact solutions of (approximate) compensation equations. Using these solutions one could study conditions under which the assumptions would be fulfilled. To some extent the model has to correspond to features of the electroweak theory. Namely we achieve a simplicity by considering a scalar field. In view of coupling constants to have proper dimensions we choose dimensionality of the space-time to be six. Really, in this case the coupling constant of interaction $g\,\phi^3$ is dimensionless and interaction $G\,\phi^4$ has constant of inverse mass squared dimension, that corresponds to the dimension of a constant of a four-fermion interaction in four-dimensional space-time.

So we introduce in the six-dimensional space-time a scalar field ϕ with initiative scale-invariant Lagrangian

$$L \,=\, \frac{1}{2}\, g^{\mu\nu}\, \frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} \,+\, \frac{g_0}{3!}\,\phi^3\,. \tag{1}$$

Let us choose the natural signature with one time and five space axes. The transition from this space-time to Euclidean six-dimensional space is accompanied by the following substitutions

$$p^2 \,\to\, -\,p_E^2\,; \qquad d^6\,p \,\to\, \imath\, d_E^6\,p\,. \tag{2}$$

It was important for us to find a model, which corresponds to the approach under consideration.

Now we start with Lagrangian (1). Evident evaluations give one-loop renormalization group equation [7] for $g^2(\mu^2)$

$$\frac{d\,g^2(\mu^2)}{dL} \,=\, -\,\frac{3\,g^4}{4\,(4\,\pi)^3}\,; \qquad L \,=\, \log\frac{\mu^2}{\Lambda_3^2}\,; \tag{3}$$

Solution of equation (3) has a form

$$g^2(\mu^2) \,=\, g_0^2\left(1 \,+\, \frac{3\,g_0^2}{4\,(4\,\pi)^3}\,\log\frac{\mu^2}{\Lambda_3^2}\right)^{-1}. \tag{4}$$

Sometimes it is convenient to use parameter $\bar{h}(\mu^2)$ defined by the following relation

$$\bar{h}(\mu^2) \,=\, \frac{3\,g^2(\mu^2)}{4\,(4\,\pi)^3} \,=\, \left(\log\frac{\mu^2}{\Lambda_g^2}\right)^{-1}; \tag{5}$$

where for transition from Λ_3^2 to Λ_g^2 we have used the standard tool analogous to that in QCD:

$$\Lambda_g^2 \,=\, \Lambda_3^2\,\exp\left(-\,\frac{4\,(4\pi)^3}{3\,g_0^2}\right).$$

Thus we get convinced, that the theory (1) is an asymptotically free one and expression (5) makes sense for $\mu^2 \,\gg\, \Lambda_g^2$.

Let us have a massless scalar field of the six-dimensional space. The initial free Lagrangian possesses scale symmetry. We shall look for a solution, which breaks this symmetry, with the aid of Bogolubov quasi-averages approach [2]. Namely according to [2] we add to the Lagrangian the following small increment

$$-\,\epsilon\,\frac{\phi^4}{4!}\,.$$

Now the scale invariance is already broken and an appearance of nonlocal terms, corresponding to a form-factor $F(p_1, p_2, p_3, p_4)$, defining range of interaction . We shall look for a solution, decreasing at momentum infinity and thus defining a region of action of the effective interaction.

Let us add to the initial free Lagrangian such a term with an interaction of the forth power and subtract the same

$$L \,=\, \frac{1}{2}\,g^{\mu\nu}\,\frac{\partial\phi}{\partial x^\mu}\,\frac{\partial\phi}{\partial x^\nu} \,-\, \frac{m^2}{2}\,\phi^2 \,-\, \frac{G}{4!}\,F\cdot\phi^4 \,-\, \frac{\epsilon}{4!}\phi^4 \,+\, \frac{g_0}{3!}\,\phi^3 \,+\, \frac{G}{4!}\,F\cdot\phi^4 \,+\, \frac{m^2}{2}\,\phi^2; \tag{6}$$

where we use abbreviated notation $-\,G\,F\cdot\phi^4$ instead of expression with a form-factor. Of course the presence of the form-factor explicitly breaks the scale invariance, so we perform a procedure "add – subtract" for a mass term as well. Let us refer the forth power term with the plus sign to the interaction Lagrangian and the same term with the minus sign we refer to the free Lagrangian.

$$\begin{aligned} L_0 \,&=\, \frac{1}{2}\,g^{\mu\nu}\,\frac{\partial\phi}{\partial x^\mu}\,\frac{\partial\phi}{\partial x^\nu} \,-\, \frac{m^2}{2}\,\phi^2 \,-\, \frac{G}{4!}\,F\cdot\,\phi^4 \,-\, \frac{\epsilon}{4!}\phi^4\,;\\ L_{int} \,&=\, \frac{g_0}{3!}\,\phi^3 \,+\, \frac{G}{4!}\,F\cdot\phi^4 \,+\, \frac{m^2}{2}\,\phi^2\,; \end{aligned} \tag{7}$$

According to the quasi-averages approach [2] the term with the plus sign has to be compensated. This means, that the new free Lagrangian leads to zero four-particle connected Green functions and as a final result contains only terms of the second power in fields. Thus performing evaluations with sign which is inherent to the term in the new free Lagrangian, we come to the compensation equation, which schematically looks in the following way: the first order term plus one-loop terms plus two-loop terms etc. Emphasize once more, that here one has to use term $+G\,\phi^4$ as an interaction Lagrangian. One has to equalize to zero the expansion obtained in such a way . This condition is an equation for function $F(p_1, p_2, p_3, p_4)$. We set $\epsilon \to 0$ after evaluations, in our case this means after compensation equations being obtained.

The equation explicitly differs from expansion in powers of interaction Lagrangian

$$L_{int} \,=\, \frac{G}{4!}\,F\,\cdot\,\phi^4\,; \tag{8}$$

in the sign of the interaction constant. In view of this note let us emphasize, that the procedure being described can be applied only to symmetry breaking terms of even powers in fields. For terms of odd powers, e.g. for three-linear ones, a fulfillment of a compensation equation leads to vanishing of connected Green function, which is defined by an interaction Lagrangian, because the two expansions in this case differ only in overall sign.

Let us construct an approximated equation for the fourth power connected Green function. First of all we choose the following kinematics: both left legs have zero momenta and the right ones have momenta p and $-p$. We restrict ourselves by terms up to two-loop ones inclusively. Namely, we have the first order term – the point; three terms of the second order – simple loops, i.e. a horizontal one and two vertical ones with permuted left legs; in the third order we have a horizontal and two vertical two-loop chains and six terms "wine glass": horizontal wine glasses having bases to the left and to the right and vertical ones with bases up and down. The number of the last terms is to be counted twice due to permutations of the left-sided momenta p and $-p$. Generally speaking, in each vertex form-factor F is present. However we can solve only a linear version of the equation, which is obtained by keeping in the equation the first and the second order terms, the two-loop horizontal chain and the wineglass with the basing to the right. We proceed to the linear equation keeping form-factor $F(p,-p,0,0) \equiv F(p^2)$ in the first order term and in right-hand vertices of the horizontal loop of the second order, of the horizontal two-loop chain and of the wineglass in the third order. Other vertices in diagrams we consider to correspond to point-like interaction in which the form-factor is changed for its value at zero ($F(0)\,=\,1$)

$$\frac{G}{4!}\,F(0)\,\phi^4 \,=\, \frac{G}{4!}\,\phi^4\,. \tag{9}$$

In vertical simple loops, which as well serve as a kernel of the integral equation, we substitute point-like vertices (9). Corresponding integrals diverge of course. In view of our search for decreasing solutions at momentum infinity for $F(p^2)$, we introduce some cut-off Λ, which existence is to be confirmed by results of a solution of the equation. In doing this we make the following substitution

$$\int_0^\infty dq^2 \,\to\, \Lambda^2\,.$$

We consider the equation in six-dimensional Euclidean space with the aid of substitutions (2). In the course of evaluations one has to perform angle integrations in six-dimensional space of functions $((p-q)^2)^{-1}$ and $\log(p-q)^2$ with powers of (pq). We have (for the logarithmic case see [8])

$$\int \frac{d\Omega_6}{p^2+q^2-2\,p\,q\,\cos\theta} \;=\; \frac{4\,\pi^3}{3}\biggl(\Theta(x-y)\Bigl(\frac{3}{4x}-\frac{y}{4x^2}\Bigr)+\Theta(y-x)\Bigl(\frac{3}{4y}-\frac{x}{4y^2}\Bigr)\biggr);$$
$$\int d\Omega_6\,\log(p^2+q^2-2\,p\,q\,\cos\theta) \;=\; \frac{\pi^3}{12}\biggl(\Theta(x-y)\Bigl(\frac{8y}{x}-\frac{y^2}{x^2}+12\,\log\,x\Bigr)+$$
$$+\,\Theta(y-x)\Bigl(\frac{8x}{y}-\frac{x^2}{y^2}+12\,\log\,y\Bigr)\biggr); \tag{10}$$
$$\int d\Omega_6\,(p\,q)\,\log(p^2+q^2-2\,p\,q\,\cos\theta) \;=\; \frac{\pi^3}{18}\biggl(\Theta(x-y)\Bigl(\frac{3y^2}{x}-6\,y-\frac{3y^3}{5x^2}\Bigr)+$$
$$+\,\Theta(y-x)\Bigl(\frac{3x^2}{y}-6\,x-\frac{3x^3}{5y^2}\Bigr)\biggr);$$
$$x\,=\,p^2\,;\qquad y\,=\,q^2\,.$$

First of all let us calculate one-loop integral keeping terms of zero and the first orders in m^2. We have for one such vertical diagram ($x = p^2$, where p is the total momentum along the loop)

$$-\,\imath\,\frac{G^2\,\pi^3}{2\,(2\pi)^6}\,\biggl(\Lambda^2\,+\,\frac{1}{3}\,x\,\log\Bigl(\frac{x}{\Lambda^2}\Bigr)\,+2\,m^2\,\log\Bigl(\frac{x}{\Lambda^2}\Bigr)\,-\,c\,x\biggr)\,; \tag{11}$$

where Λ is the square of the cut-off being mentioned and c is a constant, depending on a behaviour of the form-factor.

Let us consider the linear compensation equation, obtained in agreement with the formulated rules (see Fig. 1). The equation in this approximation has the following form

$$G\,F(p^2)\;=\;\frac{G^2}{2(4\pi)^3}\,\biggl(3\,\Lambda^2\,+\,\frac{2}{3}\,p^2\,\log\Bigl(\frac{p^2}{\Lambda^2}\Bigr)\,+4\,m^2\,\log\Bigl(\frac{p^2}{\Lambda^2}\Bigr)\,-\,2\,c\,p^2\biggr)\;-$$
$$-\,\frac{G^3}{8\,(2\pi)^9}\,\int\Bigl(\frac{1}{3}(p-q)^2\,\log\frac{(p-q)^2}{\Lambda^2}\,+2\,m^2\,\log\frac{(p-q)^2}{\Lambda^2}\,-\,c\,(p-q)^2\Bigr)\,\times \tag{12}$$
$$\times\,\frac{F(q^2)}{(q^2+m^2)^2}\,d^6q\,-\,\frac{3\,G^3\,\pi^3\,\Lambda^2}{2\,(2\pi)^{12}}\,\int\frac{F(q^2)}{(q^2+m^2)^2}\,d^6q\,.$$

Firstly let us note, that trivial solution $G\,=\,0$ is evidently possible. In view of looking for a non-trivial solution we cancel the equation by G. Performing here angle integrations by using formulas (10) we obtain the following one-dimensional integral equation

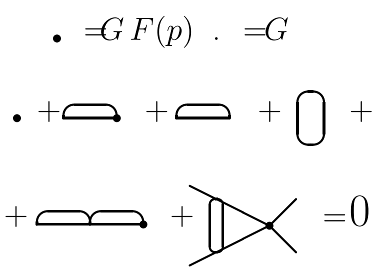

Fig. 1. The graphic representation of the linear compensation equation (18).

$$
\begin{aligned}
&F(x) = \frac{G}{2(4\pi)^3}\biggl(3\Lambda^2 + \frac{2}{3}\,x\,\log\Bigl(\frac{x}{\Lambda^2}\Bigr) + 4m^2\,\log\Bigl(\frac{x}{\Lambda^2}\Bigr) - 2cx\biggr) - \\
&-\frac{3\,G^2\Lambda^2}{4(4\pi)^6}\int_0^\infty \frac{y^2\,F(y)}{(y+m^2)^2}\,dy - \frac{G^2}{18(4\pi)^6}\biggl(-\frac{1}{20\,x^2}\int_0^x \frac{y^5\,F(y)}{(y+m^2)^2}\,dy + \\
&+\frac{3}{4x}\int_0^x \frac{y^4\,F(y)}{(y+m^2)^2}\,dy + 3\,\log\,x\int_0^x \frac{y^3\,F(y)}{(y+m^2)^2}\,dy + 3\,x\,\log\,x\int_0^x \frac{y^2\,F(y)}{(y+m^2)^2}\,dy + \\
&+3\int_x^\infty \frac{y^3\,\log\,y\,F(y)}{(y+m^2)^2}\,dy + x\int_x^\infty \frac{(4+3\log\,y)\,y^2\,F(y)}{(y+m^2)^2}\,dy + \\
&+4\int_0^x \frac{y^3\,F(y)}{(y+m^2)^2}\,dy + \frac{3\,x^2}{4}\int_x^\infty \frac{y\,F(y)}{(y+m^2)^2}\,dy - \frac{x^3}{20}\int_x^\infty \frac{F(y)}{(y+m^2)^2}\,dy\biggr) + \qquad (13)\\
&-\frac{G^2\,m^2}{12(4\pi)^6}\biggl(-\frac{1}{x^2}\int_0^x \frac{y^4\,F(y)}{(y+m^2)^2}\,dy + \frac{8}{x}\int_0^x \frac{y^3\,F(y)}{(y+m^2)^2}\,dy + 12\,\log x\int_0^x \frac{y^2\,F(y)}{(y+m^2)^2}\,dy + \\
&+12\int_x^\infty \frac{y^2\,\log\,y\,F(y)}{(y+m^2)^2}\,dy + 8\,x\int_x^\infty \frac{y\,F(y)}{(y+m^2)^2}\,dy - x^2\int_x^\infty \frac{F(y)}{(y+m^2)^2}\,dy\biggr) + \\
&+\frac{G^2}{6\,(4\pi)^6}\Bigl(\log\,\Lambda^2 + 3\,c\Bigr)\biggl(\int_0^\infty \frac{y^3\,F(y)}{(y+m^2)^2}\,dy + x\int_0^\infty \frac{y^2\,F(y)}{(y+m^2)^2}\,dy\biggr) + \\
&+\frac{G^2\,m^2}{(4\pi)^6}\,\log\,\Lambda^2\int_0^\infty \frac{y^2\,F(y)}{(y+m^2)^2}\,dy\,.
\end{aligned}
$$

A method of solution of equations of (13) type is developed in work [10]. Equation (13) is reduced to a differential one by sequential differentiations. Evident evaluation gives

$$
\begin{aligned}
&\frac{d^4}{dx^4}\biggl(x^2\frac{d^4}{dx^4}\Bigl(x^2F(x)\Bigr)\biggr) = -\,\beta\biggl(\frac{F(x)}{(x+m^2)^2} + 2m^2\Bigl(x\,\frac{d^2}{dx^2}\frac{F(x)}{(x+m^2)^2} + 3\frac{d}{dx}\frac{F(x)}{(x+m^2)^2}\Bigr)\biggr);\\
&\beta\ =\ \frac{2\,G^2}{(4\,\pi)^6}\,. \qquad (14)
\end{aligned}
$$

One easily see, that Eq. (14) can be rewritten in the form

$$
\begin{aligned}
&\biggl(\Bigl(x\frac{d}{dx} + 2\Bigr)\Bigl(x\frac{d}{dx} + 1\Bigr)\Bigl(x\frac{d}{dx}\Bigr)\Bigl(x\frac{d}{dx}\Bigr)\Bigl(x\frac{d}{dx} - 1\Bigr)\Bigl(x\frac{d}{dx} - 1\Bigr)\ \times\\
&\times\Bigl(x\frac{d}{dx} - 2\Bigr)\Bigl(x\frac{d}{dx} - 3\Bigr) + \beta\,x^2\biggr)\,F(x)\ =\ 2\,\beta\,m^2x\Bigl(F(x) + x\frac{d\,F}{dx} - x^2\frac{d^2\,F}{dx^2}\Bigr);(15)
\end{aligned}
$$

where two terms of expansion in m^2 are kept. ¿From this form of the equation we immediately conclude, that for $x \to 0$ there are eight independent asymptotes, which coefficients we denote as follows

$$
\begin{aligned}
&\frac{a_{-2}}{x^2};\quad \frac{a_{-1}}{x};\quad a_0\,;\quad a_{0l}\,\log\,x;\quad a_1\,x;\\
&a_{1l}\,x\,\log\,x;\quad a_2\,x^2;\quad a_3\,x^3\,. \qquad (16)
\end{aligned}
$$

Eight independent asymptotes at infinity are the following

$$F_k(x) \simeq x^{-3/8} \exp\Bigl(4\,(\beta\,x^2)^{1/8} \exp\bigl(\frac{\imath\,\pi\,(2k-1)}{8}\bigr)\Bigr)\,; \qquad k = 1,\,2,\,...,\,8. \tag{17}$$

Four of these asymptotes at infinity decrease exponentially ($k = 3,\,4,\,5,\,6$), and the rest four ones do increase.

Equation (15) is equivalent to the initial integral equation under definite boundary conditions being fulfilled. First of all we can use only solutions, decreasing at infinity. To obtain conditions at zero we have to substitute expression

$$F(x) = -\frac{x^2}{\beta}\,\frac{d^4}{dx^4}\Bigl(x^2\,\frac{d^4}{dx^4}\,\bigl(x^2\,F(x)\bigr)\Bigr)\,;$$

in integrals of equation (13) and perform sequential integrations by parts.

We have

$$\begin{aligned} F(x) &= F(x) - \frac{a_{-2}}{x^2} - \frac{a_{-1}}{x} - a_{0l}\,\log x - a_{1l}x\,\log x\,+ \\ &+ \frac{G\,\pi^3}{2(2\pi)^6}\,\Bigl(3\,\Lambda^2\Bigl(1 - \frac{G\,I}{2(4\,\pi)^3}\Bigr) + \frac{2\,x}{3}\log\Bigl(\frac{x}{\Lambda^2}\Bigr) - 2\,c\,x\Bigr) + x\,\Bigl(\log\,\Lambda^2 + 3\,c\Bigr)\,a_{1l}\,; \\ I &= \int_0^\infty \frac{y^2\,F(y)}{(y+m^2)^2}\,dy\,. \end{aligned} \tag{18}$$

¿From here we obtain the following condition (independently on values of Λ^2 and c)

$$a_{-2} = 0\,, \qquad a_{-1} = 0\,, \quad a_{0l} = \frac{2\,G\,m^2}{(4\pi)^3}\,,$$

$$a_{1l} = \frac{G\pi^3}{3\,(2\pi)^6} = \frac{\sqrt{2\,\beta}}{6}\,; \tag{19}$$

$$I = \frac{2(4\pi)^3}{G} = \frac{2\,\sqrt{2}}{\sqrt{\beta}}\,. \tag{20}$$

The first four conditions (19) are boundary conditions for Eq. (15). A combination of four solutions decreasing at infinity with account of these boundary conditions gives the unique solution. It can be expressed in terms of well-known special functions for case $m^2 = 0$. Indeed, let us make the following substitution in Eq. (15)

$$z = \frac{\beta\,x^2}{2^8}\,; \tag{21}$$

which reduces the equation to the canonical form of Meijer equation [11] of the eighth order

$$\begin{aligned} &\Bigl(\Bigl(z\,\frac{d}{dz} + 1\Bigr)\Bigl(z\,\frac{d}{dz} + \frac{1}{2}\Bigr)\Bigl(z\,\frac{d}{dz}\Bigr)\Bigl(z\,\frac{d}{dz}\Bigr)\Bigl(z\,\frac{d}{dz} - \frac{1}{2}\Bigr)\Bigl(z\,\frac{d}{dz} - \frac{1}{2}\Bigr)\,\times \\ &\times\Bigl(z\,\frac{d}{dz} - 1\Bigr)\Bigl(z\,\frac{d}{dz} - \frac{3}{2}\Bigr) + z\Bigr)\,F(z) = 0\,. \end{aligned} \tag{22}$$

Conditions (19) fix the solution. Firstly, four solutions, decreasing at infinity, always could be combined to set to zero three singular asymptotes at zero, i.e. to fulfill conditions $a_{-2} = a_{-1} = a_{0l} = 0$. Such property has the following Meijer function (see [11])

$$C \cdot G^{50}_{08}(\, z\,|\, 3/2,\ 1,\ 1/2,\ 1/2,\ 0,\ 0,\ -1/2,\ -1).$$

The constant is defined by the coefficient afore $\sqrt{z}\log z$. For small z this Meijer function is such [11]

$$G^{50}_{08}(z\,|\, 3/2,\ 1,\ 1/2,\ 1/2,\ 0,\ 0,\ -1/2,\ -1) \;=\; \pi \;+\; \frac{16}{3}\,\sqrt{z}\,\log\, z \;+\; \ldots . \tag{23}$$

Comparing the coefficient afore $\sqrt{z}\,\log\, z$ with (19), we obtain

$$C \;=\; \frac{\sqrt{2}}{4}\,.$$

Performing integration (see [12]), we have in accordance with definition of I (18)

$$I = \int_0^\infty F(y)\,dy = \frac{\sqrt{2}}{4}\int_0^\infty G^{50}_{08}(\beta y^2/2^8\,|\, 3/2, 1, 1/2, 1/2, 0, 0, -1/2, -1)\,dy = \frac{2\sqrt{2}}{\sqrt{\beta}}; \tag{24}$$

that perfectly agrees with condition (19).

Thus, solution

$$F(x) \;=\; \frac{\sqrt{2}}{4}\,G^{50}_{08}(\beta x^2/2^8\,|\, 3/2,\ 1,\ 1/2,\ 1/2,\ 0,\ 0,\ -1/2,\ -1)\,; \tag{25}$$

fulfills all conditions (19), and consequently the initial equation (13), which is an approximate compensation equation. This solution is a nontrivial solution, which contains dimensional parameter G, and hence it leads to the initial scale symmetry breaking. Of course as we have noted before trivial solution $F(x) \;=\; 0$ is also possible. Note, that the boundary conditions are not dependent on value of the form-factor at zero. Equality $F(0) = 1$ will serve as an additional condition in what follows.

Let us take into account terms proportional to m^2. We shall look for a correction to the solution of Eq. (15) in the following form

$$F(x) \;=\; F_0(x) \;+\; \Delta\, F(x)\,. \tag{26}$$

Substituting (26) into equation (15) we have the following equation in the first order in m^2

$$\begin{aligned}\Biggl(\Bigl(x\,\frac{d}{dx} + 2\Bigr)\Bigl(x\,\frac{d}{dx} + 1\Bigr)\Bigl(x\,\frac{d}{dx}\Bigr)\Bigl(x\,\frac{d}{dx}\Bigr)\Bigl(x\,\frac{d}{dx} - 1\Bigr)\Bigl(x\,\frac{d}{dx} - 1\Bigr)\Bigl(x\,\frac{d}{dx} - 2\Bigr) \times \\ \times \Bigl(x\,\frac{d}{dx} - 3\Bigr) + \beta\,x^2\Biggr)\,\Delta\,F(x) \;=\; 2\,\beta\,m^2\,x\,\Bigl(F_0(x) + x\,\frac{d\,F_0}{dx} - x^2\,\frac{d^2\,F_0}{dx^2}\Bigr).\end{aligned} \tag{27}$$

¿From equation (27) we can exactly define several terms of expansion of $\Delta\,F(x)$ for small x. Indeed let us consider the following expression

$$\bar{\Delta}\,F(x)\;=\;\frac{2\,m^2}{x}\left(F_0(x)\,+\,x\,\frac{d\,F_0}{dx}\;-\;x^2\,\frac{d^2\,F_0}{dx^2}\right)\;= \tag{28}$$
$$=\;2\,m^2\biggl(\frac{\pi}{2\sqrt{2}\,x}+\frac{2\,G}{3\,(4\pi)^3}\Bigl(\log\,(\sqrt{\beta}\,x)\,+4\,\gamma-\frac{23}{6}\Bigr)\,+\,\frac{\pi\sqrt{2}\,G^2}{96\,(4\pi)^6}\,x\,\log\,x+O(x)\biggr)\,;$$

where $\gamma\,=\,0.577215665...$ is the Euler constant. Substituting expression (28) into equation (27), we get convinced, that it fulfills the equation up to terms of x^3 order, because the differential operator in the left-hand side nullifies the terms presented in (28) and subsequent terms up to the indicated order. We are interested just in the presented terms (28) because they refer to the boundary conditions. Indeed expression (28) contains terms $z^{-1/2}$, $\log z$, $z^{1/2}\,\log z$, which violate their boundary conditions. Hence we are to add to expression (28) a combination of solutions of the homogeneous equation to force the boundary conditions to be fulfilled. Finally we obtain

$$\Delta\,F(x)\;=\;\bar{\Delta}\,F(x)\;-\;\frac{\pi^2\,Y}{8}\,G^{50}_{08}(\beta\,x^2/2^8\,|\,3/2,\;1,\;1/2,\;1/2,\;-1/2,\;0,\;0,\;-1\,)\;-$$
$$-\;\frac{2\,Y}{3}\,G^{50}_{08}(\beta\,x^2/2^8\,|\,3/2,\;1,\;1/2,\;0,\;0,\;1/2,\;-1/2,\;-1\,)\;-$$
$$-\,\pi\,Y\left(\gamma\,+\,\log\,2\,-\,\frac{43}{48}\right)G^{50}_{08}(\beta\,x^2/2^8\,|\,3/2,\;1,\;1/2,\;1/2,\;0,\;0,\;-1/2,\;-1\,)\,; \tag{29}$$
$$Y\;=\;\frac{G\,m^2}{2\,(4\pi)^3}\,.$$

¿From this expression we extract the exact value for $F(0)$. While doing this one has to bear in mind, that the presence of a term being proportional to $\log\,x$ at $x\,\to\,0$ is a consequence of an expansion in m^2 at $x\,\gg\,m^2$. Looking back at the corresponding evaluations we see, that for $x \to 0$ one has to change $\log\,x$ for $\log 4\,m^2$. Now we have

$$F(0)\;=\;\frac{\pi\,\sqrt{2}}{4}\;+\;Y\left(4\log Y+(16-\pi^2)\gamma+(14-\pi^2)\log 2-\frac{122}{9}+\pi^2\frac{42}{48}\right). \tag{30}$$

For $Y\,=\,0$ we obtain $F(0)\,=\,1.11072$. Condition $F(0)\,=\,1$ defines the value of Y, which is connected with mass (see (29))

$$Y\;=\;0.005789\,. \tag{31}$$

Thus the solution, which is found here, satisfies all the necessary conditions provided (31) is valid. Emphasize, that (31) defines the mass of the scalar field. Note, that the small value of (31) thoroughly justifies the account of only the first term of the expansion in m^2. We reject the second solution of condition $F(0) = 1$, which is of order of unity, due to to its inconsistence with the expansion of the solution in m^2.

Let us look at interaction Lagrangian (7). The mass term there is quite improper. To solve the problem one has to formulate a compensation equation for Green function with two scalar legs. Let us consider this equation taking into account solution (26) and threefold interaction. The compensation equation means nullification of total contribution of interaction (7) to the mass. In the first approximation the contribution of the four-fold interaction is described by the first order diagram "bubble" and that of the three-fold one is represented by simple one-loop diagram (see Fig. 2).

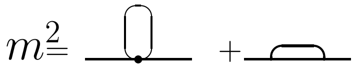

Fig. 2. Compensation equation for mass of the scalar field.

Putting momenta of the external legs to be zero, we have for "bubble" diagram just solution (26) in the vertex. As a result we obtain the following compensation equation for scalar mass

$$
\begin{aligned}
m^2 \;=&\; -\,\frac{G}{(2\,\pi)^6}\int\frac{F(q^2)\,d^6q}{q^2+m^2}\;-\;\frac{g^2}{(2\pi)^6}\int\frac{d^6q}{(q^2+m^2)^2}\;=\\
=&\; -\,\frac{G}{2\,(4\,\pi)^3}\int_0^\infty y\,dy\,(F_0(y)\;+\;\Delta F(y))\;+\;\frac{Gm^2}{2\,(4\,\pi)^3}\int_0^\infty dy\,F_0(y)\;-\\
&-\,\frac{g^2}{2\,(4\pi)^3}\int_0^\infty\frac{y^2\,dy}{(y+m^2)^2}\,; \qquad (32)
\end{aligned}
$$

Here in "bubble" diagram we perform an expansion in m^2 and take into account the zeroth and the first orders of the expansion. By direct evaluation with the aid of expressions (25, 28, 29) we obtain that the zeroth order terms is zero and the first order term is equal to $3\,m^2$. The loop, which is described by the last term in (32), quadratically diverges. Note, that in the initial theory (1) we introduce some cut-off Λ_3, which corresponds to a physical limitation of a region of applicability of the theory. As a result we have the following compensation equation for the mass provided $m \ll \Lambda_3$

$$
m^2 \;=\; 3\,m^2 \;-\; \frac{g^2}{2\,(4\pi)^3}\,\Lambda_3^2. \qquad (33)
$$

Emphasize, that for the trivial solution $G = 0$ the first term in the right-hand side of equation (33) is absent and we have a negative mass squared, i.e. a tachyon solution. For the non- trivial solution we have

$$
m^2 \;=\; \frac{g^2}{4\,(4\,\pi)^3}\,\Lambda_3^2\,. \qquad (34)
$$

It is well-known, that a scalar tachyon leads to instability for small fields. Therefore the restoration of the normal sign of the mass squared, which is achieved provided the non-trivial solution is valid, corresponds to a transition to a more stable state.

So the value of the scalar mass is defined in terms of initial parameters of the theory g and Λ_3. The value of parameter Y (31) gives the relation of the mass and of the coupling constant G of the four-fold interaction. Thus all the parameters entering into the non-trivial solution are defined in terms of the initial ones.

The final result for effective Lagrangian of the theory after the symmetry breaking occurs is the following

$$L \,=\, \frac{1}{2}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\mu} \,-\, \frac{G}{4!}\,\bar{F}(x_1,x_2,x_3,x_4)\,\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,; \tag{35}$$

where form-factor F is the solution of the compensation equation.

Grounding on the results being obtained we express a hypothesis, that in the model under consideration a nontrivial solution does exist, which breaks the initial scale invariance and leads to a spontaneous appearance of effective interaction in Lagrangian (35), acting in a restricted region of the momenta space in accordance with the value of parameter G. Effective form-factor $F(p)$ decreases exponentially with oscillations for $p^2 \,\to\, \pm\infty$, i.e. both for space-like and time-like momenta.

We start with the asymptotically free theory of a scalar field (in a six-dimensional space), and we obtain as a result the definite theory with interaction breaking scale symmetry. New dimensional parameters $G^{-1/2}$ and m are proportional to parameter Λ_3, which defines the initial asymptotically free interaction.

Of course, we base our conclusions only on exact solutions of approximate equations. However it is possible, that qualitative properties of solutions, which manifest themselves in the model problem, will be quite useful in study of problems of spontaneous symmetry breaking in more realistic cases, when there is no hope for analytic solution of corresponding equations and what is possible to apply are just numerical methods. Attractive qualitative results are the existence of relations between parameters of the problem and the natural appearance of small parameter Y (31). The essential result is connected also with the conclusion on the stability of the non-trivial solution. The estimate of non-linearity contribution, which does not lead to decisive change of properties of the solution, provides additional argument on behalf of the present approach (see work [13]).

Without any doubt a possibility of spontaneous generation of an effective interaction, containing a dimensional parameter, is of great interest for studies of problems beyond the standard theory, e.g. in application to the problem of composite Higgs field. Some considerations in connection with this aspects are described in work [5, 6].

References

[1] N.N. Bogolubov, Physica Suppl., 26 (1960) 1

[2] N.N. Bogolubov, Quasi-averages in Problems of Statistical Mechanics, Preprint JINR D-781. Dubna, 1961 (in Russian).

[3] P.W. Higgs, Physics Letters, 12 (1964) 132

[4] R.S. Chivukula, M. Narain and J. Wormsley, Rhys. Rev. D66 (2002) 010001

[5] B.A. Arbuzov, Muon g-2 anomaly and extra interaction of composite Higgs in dynamically broken electroweak theory, Preprint NPI MSU 2001-35/675, Moscow, 2001.

[6] B.A. Arbuzov, Composite 115 GeV Higgs in a Dynamically Broken Electroweak Theory, in: Proc. of the IX Conference on Supersymmetry and Unification of Fundamental Interactions, ed. D.I. Kazakov and A.V. Gladyshev, World Scientific, Singapore, 2002, p. 273

[7] N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields, N.Y., Wiley & Intersc., 1980.

[8] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev. Integrals and Series – Elementary Functions, Moscow, Nauka, 1981, p. 545 (in Russian).

[9] B.A. Arbuzov and A.T. Filippov, ŽETF, 49 (1965) 990 (in Russian).

[10] B.A. Arbuzov and A.T. Filippov, Nuovo Cimento, 38 (1965) 796

[11] H. Bateman and A. Erdelyi, Higher transcendental functions, V. 1, New York, Toronto, London, McGraw-Hill, 1953.

[12] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series – Additional Chapters, Moscow, Nauka, 1986, p. 347 – 348 (in Russian).

[13] B.A. Arbuzov, Theor. Math. Phys., 140 (2004) 1205; Teor. Mat. Fiz., 140 (2004) 367 (in Russian).