Mathematical model of the rapidly rotating gravitating or superdense neutron configurations

E.Bespalko, S.Miheev, V.Tsvetkov, I.Tsvetkov
Tver State University, Russia
170000 Russia, Tver, Zhelabova Street, 33, Email:tsvet@tversu.ru

Abstract

In the present work the mathematical model, describing distribution of the density of the superdense dense nuclear matter's of the rapidly rotating gravitating magnetic configuration (pulsar) has been built.

The task under consideration is actual in connection with studying the nature of pulsars, elucidation of the question about the equation of their superdense nuclear matter's condition and an influence of the rapid rotation on the distribution of density.

A special interest excites the question about the possible existence of bifurcation points at the parameter $\epsilon = \frac{\omega^2}{4\pi G\rho_0}$ (ω - angular speed of rotation, G - gravitational constant, ρ_0- density in the center of configuration), in which there is a branch of nonaxisymmetrical solutions for distribution of density about the rotating axis. This asymmetry of density's distribution about the rotating axis stipulates for gravitational emission of pulsar the search of which is currently under way [1]. The equation of hydrostatic equilibrium for gravitational configuration can take the following form for convenience [2-12]:

$$\Phi + \Theta - \frac{\omega^2}{2}(x^2 + y^2) = \Pi_{(PN)} + \Pi_m \qquad (1)$$

$$\Theta(\rho) = \int_0^P \frac{dP(\rho)}{\rho} \qquad (1a), \qquad \Phi = -G\int \frac{\rho(\mathbf{r}')dV'}{|\mathbf{r} - \mathbf{r}'|}, \qquad (1b)$$

where Φ-gravitational potential of the configuration, $P(\rho)$-pressure, $\Pi_{(PN)}$-the contribution of the post-Newton corrections, Π_m-magnetic stresses. The specific kind of the function $P(\rho)$ depends on the equation of nuclear matter's condition of the configuration. Further we will use numeral data for equation of nuclear substance's condition Bete-Jonson (BJ),Oppenheimer-Volkoff (OV), Raid (R) for our calculations [8].

As it will be showed in the following work the parameter of the configuration depends on the choice of the equation of the condition and subsequently they are the criteria for their choice. In particular at the meaning $\epsilon = \epsilon_k$ a sharp increase of intensity of the configuration's gravitational emission occurs at which frequency's registrations ϵ_k is defined and arguments appear in favour of the choice of the specific equation of condition.

The value of post-Newton's corrections $\mid \Pi_{(PN)} \mid$ will be equal $\left|\frac{\Phi(0)}{c^2}\right| = 1,26 \cdot 10^{-3}$ at $\rho_0 = 4 \cdot 10^{14}$ g/cm^3. That's why we can take in (1) equal to 0.

The function $\Theta(\rho)$ can be presented as a series on degrees $\left(\frac{\rho}{\rho_0} - 1\right)$

$$\Theta = \left(\frac{P_0}{\rho_0}\right)\left(\delta_0 + \delta_1\left(\frac{\rho}{\rho_0} - 1\right) + \delta_2\left(\frac{\rho}{\rho_0} - 1\right)^2 + ...\right) \qquad (2)$$

The coefficient δ_0, δ_1, δ_2 for different equation of conditions are chosen from the condition of consent (2) and (1a) and they are presented in Table 1.

Table 1

	OV	BJ	R
δ_0	2.487978	2.415387	2.378167
δ_1	0.791516	1.253114	1.177573
δ_2	-1.696462	-1.162273	-1.200594

In all the cases the mistake of approximation is less than 1% that is quit enough for the equation of nuclear mutter's condition contains many uncertainties.

The surface of configuration can be found using the condition

$$\rho(x, y, z) = 0 \tag{3}$$

that is it can be known after the equation is solved (1). But the gravitational potential Φ in (1) depends on the form of the boundary (3). That's why we'll substitute the exact configuration of the surface (3) for a pseudo surface δD, the form of which depends on the unknown parameters Z_{ijk}. These parameters are found out of the condition of the most proximity of δD and (3). The following conditions? which must be solved together with (1).

According to the works [12, 13, 14] we choose δD in the form of the perturbed ellipsoidal surface

$$\delta D: \quad x_1^2 + x_2^2 + x_3^2 + \sum_{i,j,k}^{L} Z_{i,j,k} x_1^i x_2^j x_3^k = 1 \tag{4}$$

where $x_1 = \frac{x}{a_1}, \quad x_2 = \frac{xy}{a_1}, \quad x_3 = \frac{z}{a_3}, \quad a_1, a_3$-semi-axes of the ellipsoid of rotation, which together with Z_{ijk} parameterize δD. For smooth surfaces by choice of semi-axis $a_1 \;\; a_3$ we can make coefficients Z_{ijk} small in order that use a method of decomposition of configuration's surface depending on the form into the series of Burman-Lagrange [13] on the degrees of small coefficients/ Z_{ijk}. In our case $|Z_{ijk}| \le 10^{-4}$.

The method of Burman-Lagrange's series with the use of symbolic calculations on a computer becomes a powerful method of solving very difficult sums, which couldn't be done using the old classical methods.

The conditions of proximity (3),(4) can be formed entering the function Λ:

$$\Lambda = \frac{1}{4\pi\rho_0} \int\limits_{\delta D} \rho^2 \frac{d\Omega}{\alpha^2}; \quad \alpha = e^{\frac{1}{3}} \left(1 + \frac{1 - e^2}{e^2} cos^2\Theta\right)^{\frac{1}{2}}, \quad e = \frac{a_3}{a_1} \tag{5}$$

Weighting multiplier $\frac{1}{\alpha^2}$ in (5) has been chosen for the conveniency of calculations in order the integrals in Λ and Φ to have the same structure. Evidently the parameter $\eta_{\delta D} = \Lambda^{\frac{1}{2}}$ will present a measure of mistake in our equations substituting the exact surface of the configuration for δD. In the approximation we use it increases with decrease e from 0 to $4,88 \cdot 10^{-2}$. Minimum Λ leads to the equations:

$$\Psi_{i,j,k} = \frac{\partial \Lambda}{\partial Z_{i,j,k}} = 0, \quad \Psi_1 = a_1 \frac{\partial}{\partial a_1} \Lambda(Z_{i,j,k} = 0) = 0 \tag{6}$$

$$\Psi_2 = a_3 \frac{\partial}{\partial a_3} \Lambda(Z_{i,j,k} = 0) = 0$$

Let's present the configuration's density ρ in the form of polynomial of degree P

$$\rho = \sum_{a,b,c}^{P} \rho_{a,b,c} x_1^a x_2^b x_3^c \tag{7}$$

If we chose P quite large, the with a high degree will approximate the density of the real configuration.

Calculation Φ_{abc} for the meaning $P = 4, 6, 8$ and $L = 2, 4, 6$ is impossible without using the method of symbolic calculations on a computer.

To find Φ_{abc} in the system of symbolic calculations MAPLE, we have made a program.

As the programme has turned out to be large and complex the necessity to test it. We have done two tests. A limited transition $e \to 1$ to the symmetrical case was done. It is easily calculated using an independent programme. Besides $\Delta\Phi = 4\pi G\rho$ and operating on Φ by Laplac's operator we get density multiplied by $4\pi G\rho$. Both the tests have been done successfully for different meaning $L \;\; P$. The analogous programme in the system MAPLE has been done for the functional Λ.

Will present Θ and $\Pi_{(m)}$ also in the form of polynomial of degree P on coordinates x_1^a, x_2^b, x_3^c

$$\Theta = \sum_{a,b,c}^{P} \Theta_{a,b,c} x_1^a x_2^b x_3^c; \quad \Pi_{(m)} = k_m \sum_{a,b,c}^{P} \Pi_{(m)a,b,c} x_1^a x_2^b x_3^c, \quad k_m = \frac{B_0^2}{8\pi G \rho_0 a_1^2} \tag{8}$$

In this case the system of equations (1),(6) can be presented as the system of algebraic equations relatively to ρ_{abc} and Z_{ijk}:

$$\Phi_{a,b,c} + K_0\Theta_{a,b,c} - \epsilon(\delta_{2a} - \delta_{2b})\delta_{c0} = \Pi_{(m)a,b,c} \tag{9}$$

where

$$\Psi_{i,j,k} = 0, \quad \Psi_i = 0, \quad K_0 = \frac{P_0}{2\pi G\rho_0^2 a_1^2}, \quad \epsilon = \frac{\omega^2}{4\pi G\rho_0}.$$

We shall divide the coefficients defining the structure of configuration ρ_{abc} and Z_{ijk} into symmetric and antisymmetric parts about the axis of rotation

$$\rho_{a,b,c} = \frac{\left(\frac{a+b}{2}\right)!}{\left(\frac{a}{2}\right)!\left(\frac{b}{2}\right)!}\rho_{a+b,c} + \rho_{1(ab)c}X^2 + \rho_{[ab]c}X \tag{10}$$

$$Z_{i,j,k} = \frac{\left(\frac{i+j}{2}\right)!}{\left(\frac{i}{2}\right)!\left(\frac{j}{2}\right)!}Z_{i+j,k} + \rho_{1(ij)k}X^2 + Z_{[ij]k}X$$

.

Here and hence a, b, c and i, j, k are even. And again entered values satisfy the correlation of symmetry:

$$\rho_{1(ab)c} = \rho_{1(ba)c}, \quad \rho_{[ab]c} = -\rho_{[ba]c} \tag{11}$$

$$Z_{1(ij)k} = Z_{1(ji)k}, \quad Z_{[ij]k} = -Z_{[ji]k}$$

Symmetrizing and anti-symmetrizing at the first two indices the system of equations (9) we get a new system of equations for defining $\rho_{(ab)c}$, $\rho_{[ab]c}$, $Z_{(ab)c}$, $Z_{[ab]c}$:

$$\Phi_{(ab)c} + K_0\Theta_{(ab)c} - \epsilon(\delta_{2a} - \delta_{2b})\delta_{c0} = 0, \quad \Psi_{(ij)k} = 0 \tag{11a}$$

$$\Phi_{[ab]c} + K_0\Theta_{[ab]c} = k_m\Pi_{(m)[ab]c}, \quad \Psi_{[ij]k} = 0 \tag{11b}$$

At the output (11) we have omitted the small member $\Pi_{(m),(ab)c}$, which leads only to the small correction k_m.

The coefficients in equations (11), (11b) are integrals J_{ac}, which we'll calculate numerically. The calculation of these integrals and all other calculations we'll make with accuracy 10^{-30}.

From physical point of view ϵ is a free parameter, and e is a calculated one. However, for conveniency of calculations it is more simple to consider e as a free parameter, ϵ - as the calculated parameter. The result is that all the parameters of the configuration will depend on e, ρ_0, P_0.

Substituting (10) into (9) system of equations which is a polynomial in the series of the small parameter X. That's why it's possible to use the method of decomposition in the small parameter X. In the first approximation we shall put $X = 0$ and find the meaning ρ_{ac} and Z_{ik} corresponding the figure of rotation. In this approximation we'll put $K_0 = K_{00}$, $\epsilon = \epsilon_0$.

Two-dimensional arrays unknowns in the system (11) Z_{ik}, ρ_{ac}, $K_0 = K_{00}$, $\epsilon = \epsilon_0$ identify as $y_m (m = 1, 2 \ldots N_1)$.

It is easy to find a formula for defining N_1

$$N_1 = \frac{1}{8}(P+2)(P+4) + \frac{1}{8}(L+2)(L+4) \tag{12}$$

At $P = 6$ and $L = 2$ we have $N_1 = 13$. Connection between the variables will take the form:

$$y_1 = \rho_{02}, \quad y_2 = \rho_{04}, \quad y_3 = \rho_{06}, \quad y_4 = \rho_{20}, \quad y_5 = \rho_{22}, \quad y_6 = \rho_{24}, \quad y_7 = \rho_{40},$$

$$y_8 = \rho_{42}, \quad y_9 = \rho_{60}, \quad y_{10} = Z_{02}, \quad y_{11} = Z_{20}, \quad y_{12} = K_0, \quad y_{13} = \epsilon$$

.

In this case the equations of the system (11) can take the form of algebraic equations:

$$f_l(y_m, e) = 0 \tag{13}$$

For numerical solution (13) we'll use regularized analogue of Newton's method with the parameter of regularization $\alpha = 10^{-6}$ [16]. Omitting index the component $y_m, \quad f_l$ we have the next iterative scheme:

$$y_{(y+1)(e)} = y^n(e) - \tau_n[\alpha f^2(y^{(n)}(e), e) + \widetilde{f}'(y^{(n)}(e), e) f'(y^{(n)}(e), e)]^{-1} \times$$
$$\times \widetilde{f}'(y^{(n)}(e), e) f(y^{(n)}(e), e) \qquad (14)$$

where τ_n-the step of iteration ($\Theta_0 \ll \tau_n \ll 1$), $\quad f'(y^{(n)}(e), e)$ -the matrix of Jacobi, $\widetilde{f}'(y^{(n)}(e), e)$ - transposed matrix of Jacobi.

The variable $\sqrt{f^2(y^{(n)}(e), e)} = \delta_n(e)$ is discrepancy and defines precision of the system of equations (13), which in our case is 10^{-30}.

Scheme's realization programme (14) in the same packet MAPLE as well as the above mentioned symbolic calculations Φ_{abc} Ψ_{ijk} turned out to be effective for the number of iterations for every parameter e was not more that ten. It is very important taking into consideration a slow speed of numerical calculations in the system MAPLE.

Precision of approximation of the found configuration of perturbed ellipsoidal surface δD for the case $P = 6, \quad L = 2$ increases with decrease e from 0 to $4,8 \cdot 10^{-2}$.

We put the received numerical data $y_m(e)$ into (11b). Then (11b) will contain only unknown data $\rho_{[ab]c}, \quad Z_{[ij]k}$. Just as in the previous case we'll convert these three -dimensional arrays into one-dimensional $x_m (m = 1, 2, ...N_2)$, and we'll also put $x_1 = \rho_{[20]0} = 1$. In case $P = 6 \quad L = 8 \quad N_2 = 8$.

We'll consider an influence of magnetic strains in the simplest model when two coefficients $\Pi_{(m)[20]0} = -\Pi_{(m)[02]0} = \frac{k}{2}$, k- the index of magnetic field's speed's degrease at the distance from the magnetic axis are different from 0.

We'll put $\eta_m = -\frac{1}{2} k k_m$. As the system (11b) is antisymmetric at the first two indices it will contain only add on X members. At first we'll solve this system in the linear X approach. It is well afar from critical point at the parameter e_k and respectively ϵ_k, where the definer at the unknown x_i is essentially different from 0. In the proximity of e_k it is necessary to take into consideration (11b) member of order X^3. In the linear X approach (1b) take the form:

$$\left(\sum_{p=1}^{N_2} A_l^p(y(e), e) x_p \right) X = \eta_m \delta_{l1} \qquad (15)$$

The variables x_p, X are solved analytically as function η_m at the set numerals $y(e)$ and e with the help of MAPLE.

The meanings e_k are found out of the condition $X(e \to e_k) \to \infty$. The calculations are presented in table 2.

Table 2

	OV	BJ	R
e_k	0.58673	0.59207	0.59135
ε_k	0.09675	0.09684	0.09695

It is necessary to note the proximity of the meanings e_k and ϵ_k for the three equations of condition. (OV, BJ ,R).

The linear approach can be applied in the region $|e - e_k| \gg \eta_m^{\frac{2}{3}}$. At $|e - e_k| \leq \eta_m^{\frac{2}{3}}$ the system (15) is substituted for cubical about X system

$$(A_l^p) X + (B_l^{p+r} x_p x_t x_r) X^3 = \eta_m \delta_{l1}; \quad l, p, t, r = 1, 2...N_2 \qquad (16)$$

Here the summing at the repeated indices is done.

Taking $l \neq 1$ into (15) we'll find solutions $x_p = x_p(e)$, which preserve their meanings and at $e = e_k$. At $|e - e_k| \gg \eta_m^{\frac{2}{3}}$ taking $l = 1$ into (12) we'll find $X = X(e, \eta_m)$, besides $X \simeq \eta_m$.

At $|e - e_k| \leq \eta_m^{\frac{2}{3}}$ we'll put $X = X(e, \eta_m)$ into (16) and then we have a cubical equation in this field for X.

$$\alpha(e - e_k) X + \beta X^3 = \eta_m \qquad (17)$$

where α and β are constant coefficients, which meaning are presented in table 3.

Table 3

	OV	BJ	R
α	0.62596	0.41016	0.42876
β	10.8375	2.3442	2.7674

In the critical point $e = e_k$, $X_k = X(e_k) = \eta_m^{\frac{1}{3}}/\beta$, is asymmetry of density's distribution is much more than the linear approach. Note that for pulsar $\eta_m \sim 10^{-9} - 10^{-12}$.

If $\eta_m = 0$, then at $e > e_k$ $(\epsilon < \epsilon_k)$ (17) has one decision $X = 0$, that corresponds to axial symmetrical configuration. But at $e < e_k$ $(\epsilon > \epsilon_k)$ there will be 2 roots $X = 0$ (symmetric solution) and $X = \sqrt{\frac{\beta}{\alpha}(e_k - e)}$ (asymmetric solution). A well-know dynamic violation of axial symmetry for nonlinear equations in configuration's density distribution occurs.

Dependence $\alpha X = X(e)$ are presented in table 4.

Table 4

e	X/η_m (BJ)	e	X/η_m (OV)
1	5.297	1	3.38238
0.95	6.08079	0.95	3.87927
0.9	7.13011	0.9	4.54211
0.85	8.60169	0.85	5.46700
0.8	10.80368	0.8	6.84062
0.75	14.4357	0.75	9.07916
0.7	21.49619	0.7	13.33520
0.65	40.88923	0.65	24.40889
0.6	306.19759	0.6	119.48844
0.59208	296239.23811	0.587	5983.47999
0.5920717750	483161352.66966	0.58677	43234.55271
0.59207177	52902983654.17490	0.5867334	4567205.28503
0.59207176997	151567305446.40869	0.5867330528	617880443.66757
0.59207176996	400620145034.52280	0.5867330502	-110332016807.96199

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