Nonlinear quantum field theory on the classical backgound [1]

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Abstract

New scheme of quantization is presented. This method of Bogoliubov group variables is applicable for boson fields with nontrivial classical background.

In present article we offer the method of quantization of nonlinear boson fields. We quantize in a vicinity a nontrivial classical component by means of the Bogoliubov group variables. This scheme is applicable for any system where the following sypmplectic forms exists

$$\omega(f(x), g(x)) = \int_D (f_n(x)g(x) - f(x)g_n(x))dx \tag{1}$$

Here D is some space-like surface and $f_n(x)$ is normal derivative of $f(x)$ in the surface D. (It is possible if we know the representation of symmetry group of the system.)

Boson field operator always contains a classical boson field, because canonical transformations of a boson field always extract the c-numerical component.

Let us remind (von Neumann theorem), that for any representation of the operator α, $\alpha^{'}$ (and vectors of states) there is a unitary operator U, $(U^+U = 1)$ such that $\alpha^{'} = U\alpha U^+$. Let us introduce the operator

$$\hat{S} = e^{\lambda \sum_f \hat{a}_f^+ b_f - \hat{a}_f b^*{}_f} = e^{\lambda T},$$

Exponent means series

$$\hat{S} = \lambda\big((\hat{a}_f^+ b_f - \hat{a}_f b^*{}_f) + \frac{1}{2!}(\hat{a}_f^+ b_f - \hat{a}_f b^*{}_f)^2 + \frac{1}{3!}(\hat{a}_f^+ b_f - \hat{a}_f b^*{}_f)^3 + ...\big),$$

where $\hat{a}_f$ is any arbitrary operator, b_f- c-number. Hence

$$\hat{S}^+ = e^{\lambda \sum \hat{a}_f b^*{}_f - \hat{a}_f^+ b_f} = e^{-\lambda \sum \hat{a}_f^+ b_f - \hat{a}_f b^*{}_f} = e^{-\lambda T},$$

and

$$\hat{S}^+\hat{S} = e^{\lambda T} e^{-\lambda T} = 1,$$

The most general form of canonical transformation looks like:

$$\hat{a}_f(\lambda) = \hat{S}^+(\lambda)\hat{a}_f\hat{S}^+(\lambda), \qquad [\hat{a^+}_f(\lambda), \hat{a}_f(\lambda)] = [\hat{a^+}_f, \hat{a}_f].$$

Let's differentiate this equation on parameter λ

$$\frac{d\hat{a}_f(\lambda)}{d\lambda} = \hat{S}^+(\lambda)T\hat{a}_f\hat{S}^+(\lambda) - \hat{S}^+(\lambda)\hat{a}_f T\hat{S}^+(\lambda) = \hat{S}^+(\lambda)[T\hat{a}_f]\hat{S}^+(\lambda) = b_f,$$

That is

$$\hat{a}_f(\lambda) = \hat{a}_f + \lambda b_f.$$

It means that we separate the c-numerical components by means of a nonidentical canonical transformation. For example,

$$\hat{a}_f(1) = \hat{a}_f + b_f = \hat{A}_f,$$

and

$$[\hat{A^+}_f, \hat{A}_f] = [\hat{a^+}_f, \hat{a}_f].$$

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Free fields only have no classical component. Hamiltonian

$$H=\sum_f \omega_f \hat{A}^+\hat{A}_f=\sum_f \omega_f a^+{}_f\hat{a}_f+\sum_f \omega_f(a_f b^*{}_f+a^+{}_f b_f)+\sum_f \omega_f b^*{}_f b_f.$$

Let's denote —0 ¿ to be vacuum of the operator $a_f|0>=0$ The expectation value of Hamiltonian is

$$<0|H|0>=\sum_f \omega_f b^*{}_f b_f.$$

If $b_f=0$ than the expectation value on vacuum is minimal. Suppose that the field is not free (interaction V). In this case $b_f\neq 0$ is makes a vacuum to be the minimum:

$$<0|V|0>+\sum_f \omega_f b^*{}_f b_f=0$$

Therefore quantization of any nonlinear physical system should include quantization of a boson field in a vicinity of a classical background.

If those fields are small they could be considered zero-like. In this case all the effects are described in the framework of usual perturbation theory. However even in a quantum electrodynamics we are not able to neglect by a classical field, because this approach leads to infra-red divergencies. Those difficulties are removed beyond the framework of the perturbation theory. For gravity and extended particles it is impossible to consider a classical field to be small. So we have to describe the effect of separation of a classical boson field like the most significant.

All theories of quantization on a classical background meet two difficulties. First (and main) difficulty is a problem of conservation laws. Explicit separation of the classical components leads to violation of the conservation laws. Let us write down explicit separation of a classical field in the form

$$\hat{\phi}(x)=\phi_0(x)\hat{E}+\hat{\phi}_1(x).$$

Many systems have continuous spatial symmetry, and each general theory should be relativistic invariant. Suppose we know the representation of symmetry group, which describes the system. The operator $\hat{S}$ describes transformation of this group and the classical field does not change:

$$\hat{S}\hat{\phi}(x)\hat{S}^+=\phi_0(x)+\hat{S}\hat{\phi}_1(x)\hat{S}^+,$$

that is

$$\hat{\phi}(x^{'})=\phi_0(x)+\hat{\phi}_1(x^{'}).$$

It is clear, that in such case the classical field can be considered as external one. It means that upon shift a point x the quantum field varies, and classical remains former. Hence explicit separation of a classical field breaks the invariance of a system with respect to a given symmetry group. It means that conservation laws are violated. This point is inadmissible because it contradicts basic physical principles.

The zero-mode problem is the second difficulty of quantization on a classical background. Suppose that

$$E_0^{cl}=V(a).$$

is the ground state of one-dimensional particle in the potential $V(x)$. The minimum of potential is at the point $x=a$. Particle fluctuates in a vicinity of this energy

$$E_0^{qu}=E_0^{cl}+\Delta_0=V(a)+\Delta_0,$$

where Δ_0 is a quantum correction to energy. In a vicinity of a minimum it is possible to consider the potential to be approximately harmonic:

$$V(x)=V(a)+\frac{1}{2!}\omega^2(x-a)^2+\frac{1}{3!}\lambda_3(x-a)^3+\frac{1}{4!}\lambda_4(x-a)^4...$$

In this case the basic state is

$$E_0 = V(a) + \frac{\hbar\omega}{2} + O(\lambda_k).$$

and the wave functions are located in a vicinity of the point $x = a$. Suppose that the potential has translation symmetry, that is $V(x) = V$, so $\omega = 0$ (zero mode). In this case each point x is the classical solution, but the quantum wave functions are not located in its vicinity. Connection of classical and quantum pictures thus is broken, and divergencies appear when $\omega = 0$ occurs in a denominator. This elementary example shows that the attempts to quantize of system with continuous symmetries on a classical background lead to a zero-mode problem.

To bypass these problems we propose to use the idea of Bogoliubov [I]. N.N.Bogoliubov has proposed to quantize in terms of new variables. Those variables have a sense of parameters of a symmetry group of a system and they are cyclic. The integrals of motion in terms of new variable turn out to be generators of the group and, hence, commute with Hamiltonian. Thus explicit and exact performance of the conservation laws in any order of the perturbation theory is achieved. New variable, which can be associated with collective coordinates, removes a zero-mode problem.

Bogoliubov transformations are widely used in quantum field theory. However, if the group of invariance of a system includes transformation of time, there are some difficulties, connected with the following. It is necessary to know equations of motion to get the expression for Hamiltonian as the generator of time translation. From the other hand, it is necessary to know the expression for Hamiltonian to write the equations of motion. This difficulty restricts the applicability of Bogoliubov group variables.

In this work we propose the quantization in a new way, which will allow one to use Bogoliubov group variables for systems having an arbitrary symmetry group.

The development of a scheme of quantization occurs in three basic stages.

I. At the first stage we pass to new variables - Bogoliubov group variables. All quantities are rewritten in terms of new variables.

Let the variable $x^{'}$ to be related to x by group transformation:

$$x' = F(x, a), \qquad x'' = F\left(F(x, a), b\right) = F(x, c), \qquad c = \varphi\left(a, b\right).$$

Variations of coordinates at a variation of parameters of the group a are

$$\left(\delta x'\right)^i = \xi^i_s(x') B^s_p(a) \delta a^p,$$

where $i = 0, 1, 2, 3$ is the number of coordinate, $p = 1..., r$, and k is the number of generators of the group. The group properties of transformation are defined by tensor $B^s_p(a)$.

Let us define Bogoliubov transformation as follows:

$$f(x) = gv(x^{'}) + u(x^{'}),$$

dimensionless parameter g is supposed to be large, and $(u(x^{'}), a)$ are independent new variables (Bogoliubov group variables). Explicitly separated large component depends on $x^{'}$ as well as a small part. Thus we restore the invariance with respect to transformation group, which was broken by explicit separation of the classical component, as it was pointed out in the introduction. However consideration of variable τ as an independent leads to the fact that the right-hand side of our equation contains now by k variables more than the left one. In order to have the same number of variables we shall impose on $u(x^{'})$ some invariant conditions.

Let us make it as follows: we choose some functions $N^a(x^{'})$ (the number of functions is equal to the number of group parameters) and we shall impose on function $u(x)$ the following conditions:

$$\omega(N^a, u) = 0.$$

Let us remind that $\omega(N^a, u)$ is the invariant symplectic form (1).

Use of this condition allows one to obtain the equations which determine group variable as functionals of initial $f(x)$ and $f_n(x)$ on D in the differential form:

$$\frac{\delta\tau^a}{\delta f(x)} = -\frac{1}{g} Q^a_b \tilde{N}^b_n(x^{'}), \qquad \frac{\delta\tau^a}{\delta f_n(x)} = \frac{1}{g} Q^a_b \tilde{N}^b(x^{'}),$$

where Q^a_b is solution of the equation:

$$Q^a_b = \delta^a_b - \frac{1}{g} R^a_c Q^c_b.$$

$\tilde{N}^a$ is the linear combination of N^a, such that $\omega(\tilde{N}^a, v_b) = 0$ is respected, R^a_c is c-number obtained with the help $v(x^{'})$ and $u(x^{'})$.

II. At the second stage we develop the perturbation theory. All integrals of motions are expressed in terms of new variables. Analysis of expressions gives us conditions, at which application of perturbation theory is possible, namely, the classical part owes to satisfy some equations. These equations turn out to coincide with Euler-Lagrange equations. However this result is not trivial, it was obtained not as a consequence of the variational principle, but as a condition of development of a perturbation scheme for our system in terms of new variables, while in order to get these equations we did not need to know the Hamiltonian structure as the generator of time translations.

So, at the second stage we obtain the equations of motion for classical components.

It is made as follows: Having an explicit form of expression for new variable as functional from old variable on D it is possible to define the operators $\hat{q}(x)$ and $\hat{p}(x)$:

$$\hat{p}(x) = \frac{1}{\sqrt{2}}\Big(f_n(x) + i\frac{\delta}{\delta f(x)}\Big), \qquad \hat{q}(x) = \frac{1}{\sqrt{2}}\Big(f(x) - i\frac{\delta}{\delta f_n(x)}\Big).$$

they are hermitian in space, where the scalar product looks like

$$< F_1|F_2 >= \int Df Df_n F_1[f, f_n] F_2[f, f_n],$$

and also satisfy formal commutation relations

$$[\hat{q}(x), \hat{p}(x^{'})] = i\delta(x - x^{'}).$$

Therefore we can treat $\hat{q}(x)$ and $\hat{p}(x)$ as operators of coordinate and momentum of a field oscillators and develop a scheme of secondary quantization. However another pair of self-conjugated operators exists. They are constructed from $f_n(x)$, $f(x)$, $\frac{\delta}{\delta f(x)}$ and $\frac{\delta}{\delta f_n(x)}$, and satisfy the same commutation relation. Thus, number of possible field states increases by a factor of two and requires a reduction.

Therefore we do it in the following way:

we shall develop a scheme of the perturbation theory in terms of Bogoliubov variables despite the appearance of extra states. After that we make a reduction of states, that is this reduction will be the dynamical scheme.

Now we have quantum operators of coordinate and momentum (and they are expressed in terms of new variable). Now we can perform quantization of a system by the following replacement:

$$f(x^{'}) \longrightarrow \hat{q}(x), \qquad f_n(x^{'}) \longrightarrow \hat{p}(x).$$

In terms of the new variable $\hat{q}(x)$ and $\hat{p}(x)$ are the series in inverse powers of a coupling constant. Hence the integrals of motion of a system are series:

$$O = g^2 O_{-2} + g O_{-1} + O_0 + \frac{1}{g} O_1 + ...$$

Here O_{-2} are C-numbers, the operators O_{-1} are linear with respect to $u(x^{'})$, $u_n(x^{'})$, $\frac{\partial}{\partial u(x^{'})}$, $\frac{\partial}{\partial u_n(x^{'})}$. Since there are no normalized eigenvectors of such operators for correct development of perturbation theory it is necessary that operators were identically equal to zero. This requirements are performed, if classical component satisfies equations of motion on D (at a certain boundary conditions). Then the operators O_{-1} are equal to zero.

Therefore further we shall consider the classical part to be solution of Cauchy problem with boundary conditions on D.

III. The third stage, construction of a system state space .

Above we have mentioned, that the number of possible field states increases by a factor of two in comparison with real situation, and consequently the reduction of the number of states is necessary.

First of all we shall analyse the number of independent variables. The initial number of independent variables was ∞. After definitions of Bogoliubov group variables (they, we remind, are considered as independent) this number became $\infty + k$. This number increased by a factor of two because of a way of definition of the creation-annihilation operators: $(\infty+k)*2 = 2*\infty+2*k$. Constraints which are entered together with definition of Bogoliubov transformation, reduce the number of independent variable on k (number of group parameters). And at present moment the number of possible field states is $2*\infty+k$.

Let us separate from a field variable $u(x^{'})$ k some variable r_a, which has no physical sense and is related to the method of constrution of the perturbation theory. Then number the of states is $2*\infty$ and it is possible to describe the field by means of variable $w(x^{'})$ which is defined as follows:

$$u(x^{'}) = w(x^{'}) + \tilde{N}^a(x^{'})r_a, \qquad u_n(x^{'}) = w_n(x^{'}) + \tilde{N}^a_n(x^{'})r_a.$$

Necessary reduction of the number of states can be made as follows: let's assume that a state of a field is defined by functionals $w(x^{'})$ and $w_n(x^{'})$, for which $\frac{\delta}{\delta w(x')}$ and $\frac{\delta}{\delta w_n(x')}$ act by the following way:

$$\frac{\delta}{\delta w(x')} \longrightarrow \frac{\delta}{\delta w(x')} - iw_n(x^{'}), \qquad \frac{\delta}{\delta w_n(x')} \longrightarrow -iw(x^{'}).$$

(It can be achieved, for example, by the introduction of holomorphic representation in a certain isomorphically connected space). After reductions we have the following independent variables:

(k of group parameters) + (k variable r_a) +

($\infty - k$ - dimensional space of functions w)

Variables r_a have no physical sense. They have appeared as the rest of a reduction of states space in terms of Bogoliubov group variables. Separation of these variables is related to structure of integrals of motion in zeroth order and also is dynamical by nature.

At this stage we achieve correct number of degrees of freedom. In the process of reduction we obtain equations of motion for quantum correction to our field (the equations appear to be nonlinear). Also at this stage the explicit expressions for creation-annihilation operators of quantum of a field are obtained as well as the spectrum of frequencies. Besides we obtain the expressions for integrals of motion as generators of the corresponding symmetry transformations:

$$O_0 = i\Im^\alpha \frac{\partial}{\partial \tau^\alpha},$$

In particular, Hamiltonian is the generator of time shift. Let us note, that equations of motion of a classical part and explicit form of Hamiltonian are obtained independently to overcome the basic difficulty of the description of non-stationary systems, which was described in introduction.

Also in zeroth order we shall obtain Heisenberg equation in terms of new variables.

The scheme proposed above has been applied to different systems. Most complicated example is interacting gravitational and scalar fields. We found that the classical part of scalar field Φ is the solution of equation

$$\Phi_{nn} = a\sqrt{F}\left(-F^{\lambda\mu}\Phi_{\lambda\mu} + m^2\Phi\right).$$

Here $F^{\lambda\mu}$ is the classical part of gravitational filed that should satisfy the equations

$$F_{stn} = \frac{2a}{\sqrt{F}}\left(F_{stn} - \frac{1}{2}F_n F_{st}\right),$$

$$\begin{aligned} F^{st}_{nn} = \frac{a}{2\sqrt{F}}\left(F_{lkn}F^{lk}_n - \frac{1}{2}F^2_n\right)F^{st} - \frac{2a}{\sqrt{F}}\left(F^{st}_n F^{kl}_n F_{stn} - \frac{1}{2}F_n F^{st}_n\right) - \\ -a\sqrt{F}\left(R^{st} - \frac{1}{2}F^{st}R\right) - \sqrt{F}\left(F^{sl}c^t_{;l} - F^{st}c^l_{;l}\right) + \\ +a\sqrt{F}\left(\frac{1}{2}\left(-\frac{1}{a^2}\Phi^2_0 + F^{sp}\Phi_s\Phi_p - m^2\Phi^2\right)F^{sp} - \Phi^s\Phi^p + \frac{F^{st}}{F}\Phi_n\right), \end{aligned} \tag{2}$$

$$\frac{1}{\sqrt{F}}\left(F_n^{st}F_{stn}-\frac{1}{2}F_n^2\right)-\sqrt{F}R(F)=\frac{1}{2\sqrt{F}}\Phi_n^2-\frac{\sqrt{F}}{2}\left(F^{st}\Phi_s\phi_t-m^2\Phi^2\right),$$
$$F^{sl}_{;l}=\frac{\sqrt{F}}{2}F^{st}\Phi_n\Phi_t$$
and that we can treat as metric.

Quantum corrections w_{st} for gravity satisfy the equations

$$w_{st}(\vec{x})=\frac{2}{\sqrt{F}}\left(w_{st}(\vec{x})-\frac{1}{2}F_{st}w(\vec{x})\right),$$

$$\omega_m^2 w^{st}(\vec{x})=2\sqrt{F}\left(Rw^{st}(\vec{x})-\frac{1}{2}A^{plst}\varphi_{mpl}(\vec{x})\right)w_{st}(\vec{x})-$$
$$-2\sqrt{F}\left(R^{st}(w(\vec{x}))-\frac{1}{2}R(w(\vec{x}))F^{st}\right)w_{st}(\vec{x})-Y(w(\vec{x})).$$
and quantum fart μ for scalar field is the solution of equation:
$$\tilde{\omega}^2\mu=-a\sqrt{F}\left(-F^{st}\mu_{;st}+m^2\mu\right).$$

As an example we consider quantization on the background on flat metric

$$\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$
when $F_{st}=\delta_{st}$. Quantum correction for gravity w_{st} looks like

$$w_{st}=\begin{pmatrix}w_{22}&w_{23}\\w_{23}&-w_{22}\end{pmatrix}$$

Suppose that quantum correction depends on only one space coordinate (say, x) only. In this case the classical part of scalar field is the solution of

$$\Box\Phi+m^2\Phi=0,$$

and equation for quantum correction has the form

$$\Box\mu+m^2\mu=0.$$

Equations of motion for quantum part of gravitational filed look like

$$\frac{\partial^2 w_{st}}{\partial t^2}=\frac{\partial^2 w_{st}}{\partial x^2},\qquad\frac{\partial^2 w_{st}}{\partial x^2}=\left(\Phi_{nn}\Phi-m^2\Phi^2\right)w_{st}.$$
It means that quantum part of gravitational filed is the flat double-polarized wave. Other results can be found in [3].

So we found the equation of motion of a classical part, the equation of motion of the quantum correction, the energy spectrum, integrals of motion, Heisenberg equations, and the explicit form of the field operator in terms of new variables. These results allow to assert that our theory gives the complete description of a system of bosonic fields, quantized on the classical background. The theory guarantees exact performance of the conservation laws in any order of perturbation theory and also allows to avoid a zero-mode problem.

References

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