Structures of the expectation values of flavor neutrino charges in quantum field theory

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Abstract

As a quantum-field-theoretical approach to various processes related with neutrinos, we examine the expectation values of the flavor neutrino charges at a time x^0 with respect to the state $\left|\Psi(x^0)\right\rangle$, which is taken so as to coincide with the neutrino source state such as a charged pion at a time x_I^0 $(< x^0)$. A remark is given on the way how to construct one-particle flavor neutrino state with definite 3-momentum and helicity.

1 Introduction

In the quantum field theory[1], the expectation value of a physical observable $F(x)$ at a space-time point $x = (\mathbf{x}, x^0)$ with respect to the state $\left|\Psi(x^0)\right\rangle$ is given by, in the interaction representation,

$$\left\langle\Psi(x^0)\right| F(x) \left|\Psi(x^0)\right\rangle = \left\langle\Psi(x_I^0)\right| S^{-1}(x^0, x_I^0)F(x)S(x^0, x_I^0) \left|\Psi(x_I^0)\right\rangle, \tag{1}$$

$$S(x^0, x_I^0) = \sum_{m=0}(-i)^m \int_{x_I^0}^{x^0} d^4y_1 \int_{x_I^0}^{y_1^0} d^4y_2 \cdots \int_{x_I^0}^{y_{m-1}^0} d^4y_m H_{int}(y_1)\cdots H_{int}(y_m). \tag{2}$$

In the following we consider the expectation values of the flavor neutrino charges $N_\rho(x^0), \rho =$ the flavor suffix $(e, \mu, \cdots)$, with respect to the state $\left|\Psi(x^0)\right\rangle$ which coincides with the neutrino-source state $\left|\Psi(x_I^0)\right\rangle$ such as a charged pion at the time x_I^0 $(< x^0)$. We point out that, from those expectation values, we obtain a unified way for treating various neutrino processes, such as weak decay probabilities of neutrino-source particles and the neutrino oscillation, and a suggestion how to construct the flavor neutrino state with definite 3-momentum $\mathbf{k}$ and helicity.

The total Lagrangian $\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_{int}(x)$ related to neutrinos is taken to be

$$\mathcal{L}_0(x) = -\bar{\nu}_F(x)[\not\partial + M]\nu_F(x), \qquad M^\dagger = M, \tag{3}$$

$$\mathcal{L}_{int}(x) = -[\bar{\nu}_F(x)J_F(x) + \bar{J}_F(x)\nu_F(x)] = -H_{int}(x); \tag{4}$$

here, $\nu_F(x)$ is a set of the flavor neutrino fields $\nu_\rho(x)$'s, and a set of the mass eigenfields $\nu_j(x)$'s is defined by $\nu_F(x) = Z^{1/2}\nu_M(x),\ {Z^{1/2}}^\dagger M Z^{1/2} = M_{diag}$, where

$$\mathrm{diag}(M_{diag}) = (m_1, m_2, \cdots) \quad , {Z^{1/2}}^\dagger Z^{1/2} = I,\ Z^{1/2} = [Z_{\rho j}^{1/2}], \tag{5}$$

$$\nu_F(x) = (\nu_e(x), \nu_\mu(x), \cdots)^T, \quad \nu_M(x) = (\nu_1(x), \nu_2(x), \cdots)^T; \tag{6}$$

the source function $J_F(x) = (J_e(x), J_\mu(x), \cdots)^T$ is for simplicity assumed to include no neutrino fields, and $\mathcal{L}_{int}(x)$ is the local Fermi interaction obtained in accordance with the electro-weak unified theory. We employ Greek and Roman indices, $(\rho, \sigma, \lambda, \cdots)$ and $(i, j, l, \cdots)$, to represent the flavor and the mass degrees of freedom, respectively.

It is noted that $F^H(x) \equiv S^{-1}(x^0, x_I^0)F(x)S(x^0, x_I^0)$ appearing in (1) is a quantity in Heisenberg representation, which is taken so as to coincide with the interaction representation at the time x_I^0. Hereafter we denote quantities in Heisenberg and the interaction representations by attaching and omitting the prefix H, respectively. We see that $(\not\partial + M_{diag})\nu_M(x) = 0$ leads to $(\not\partial + M)\nu_F(x) = 0$ and $(\not\partial + M)\nu_F^H(x) = -J_F^H(x)$. The integral of the latter is easily expressed in the form of so-called Yang-Feldman equation[2] when the field-mixing exists. Its implication in the neutrino problem will be discussed elsewhere. In the following we consider directly the perturbative expansion of $S(x^0, x_I^0)$ with respect to the weak interaction

$$H_{int}(x) = \sum_\sigma \frac{G_F}{\sqrt{2}} i\bar{\nu}_\sigma(x)v^a l_\sigma(x)j_a^{had}(x)^\dagger + H.c., \tag{7}$$

where $v^a = \gamma^a(1+\gamma_5)$; G_F is Fermi coupling constant. We employ Kramers representation[3] of γ^a-matrices, $\overrightarrow{\gamma} = -\rho_y \otimes \overrightarrow{\sigma}$, $\gamma^4 = \rho_x \otimes I$, $\gamma_5 = -\rho_z \otimes I$. The flavor neutrino charge N_ρ, the expectation value of which is to be examined, is defined by

$$N_\rho(x) \equiv : i \int d^3x j^4_{(\rho)}(\mathbf{x}, x^0) :, \tag{8}$$

where the 4-vector current of ν_ρ-field is given by $j^a_{(\rho)}(x) = -i\bar{\nu}_\rho(x)\gamma^a\nu_\rho(x)$. The symbol : : represents the normal product with respect to the momentum-helicity creation and annihilation operators of the $\nu_M(x)$ field, which is expanded as

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{1}{\sqrt{V}} \left[\alpha_j(kr)u_j(kr)e^{i(kx)} + \beta_j^\dagger(kr)v_j(kr)e^{-i(kx)}\right]; \tag{9}$$

here $(kx) = \mathbf{k}\cdot\mathbf{x} - \omega_j(k)x^0$ with $\omega_j(k) = \sqrt{\mathbf{k}^2 + m_j^2}$; "$r$" represents the helicity, $r = \uparrow\downarrow$; $(i\not{k} + m_j)u_j(kr) = 0$, $(-i\not{k} + m_j)v_j(kr) = 0$; α_j, β_j and their Hermitian conjugates satisfy $\{\alpha_j(kr), \alpha_i^\dagger(k'r')\} = \{\beta_j(kr), \beta_i^\dagger(k'r')\} = \delta_{ji}\delta_{rr'}\delta(\mathbf{k}, \mathbf{k}')$, others $= 0$.

2 Expectation values of flavor neutrino charges

We investigate the structure of

$$\langle A(x_I^0)|\, S^{-1}(x^0, x_I^0)N_\rho(x^0)S(x^0, x_I^0)\,|A(x_I^0)\rangle = \langle A(x_I^0)|\, N_\rho^H(x^0)\,|A(x_I^0)\rangle, \tag{10}$$

where $|A(x_I^0)\rangle$ $(x_I^0 \le x^0)$ is a hadronic state with no neutrinos and plays a role of a neutrino source, such as one-pion state. For simplicity, we consider the case where an A-particle is a (pseudo-)scalar one with decay modes caused by (7). The contributions of G_F 0-th and first orders are easily seen to vanish. There are two kinds of G_F^2-order connected contributions, expressed as

$$\begin{aligned} &\langle A(x_I^0)|\, N_\rho^H(x^0)\,|A(x_I^0)\rangle_{con(I)} \\ &\qquad \equiv \langle A(x_I^0)| \int_{x_I^0}^{x^0} d^4z \int_{x_I^0}^{x^0} d^4y H_{int}(z)N_\rho(x^0)H_{int}(y)\,|A(x_I^0)\rangle_{con}, \end{aligned} \tag{11}$$

$$\begin{aligned} &\langle A(x_I^0)|\, N_\rho^H(x^0)\,|A(x_I^0)\rangle_{con(II)} \equiv \langle A(x_I^0)|\, i^2 \int_{x_I^0}^{x^0} d^4y \int_{x_I^0}^{y^0} d^4z \\ &\qquad \times \left[H_{int}(z)H_{int}(y)N_\rho(x^0) + N_\rho(x^0)H_{int}(y)H_{int}(z)\right]\,|A(x_I^0)\rangle_{con}. \end{aligned} \tag{12}$$

We examine dominant contributions coming from diagrammatically lower configurations in intermediate states. Those contributions come from (11) and, in the case of a positively charged A-particle, are expressed as

$$\begin{aligned} \langle N_\rho(x^0); A^+(x_I^0)\rangle_{con} \equiv &\int d^3x \int_{x_I^0}^{x^0} d^4z \int_{x_I^0}^{x^0} d^4y \langle A^+(x_I^0)| \sum_{\lambda\sigma} \bar{J}_\lambda(z)\gamma^4\mathcal{S}^{(+)}_{\rho\lambda}(x-z)^\dagger \\ &\times \cdot\mathcal{S}^{(+)}_{\rho\sigma}(x-y)J_\sigma(y)\,|A^+(x_I^0)\rangle_{con}; \quad \langle 0|\,\nu_\rho(x)\bar{\nu}_\sigma(y)\,|0\rangle = -i\mathcal{S}^{(+)}_{\rho\sigma}(x-y). \end{aligned} \tag{13}$$

The state $|A^\dagger(x_I^0)\rangle$ is now taken to be a plane-wave, i.e. $|A^\dagger(p, x_I^0)\rangle = \alpha_A^\dagger(p)\,|0\rangle e^{iE_A(p)x_I^0}$ with $p^0 = E_A(p) = \sqrt{\mathbf{p} + m_A^2}$, $[\alpha_A(p), \alpha_A^\dagger(p')] = \delta(\mathbf{p}, \mathbf{p}')$. RHS of (13) includes a hadronic part, expressed as

$$\begin{aligned} &\langle A^+(p, x_I^0)|\, j_a^{had}(z)\cdot j_b^{had}(y)^\dagger\,|A^+(p, x_I^0)\rangle = \langle A^+(p, x_I^0)|\, j_a^{had}(z)\Big\{\, |0\rangle\langle 0| \\ &\quad + \sum |\text{1-particle}\rangle\langle\text{1-particle}| + \text{higher configurations}\Big\} j_b^{had}(y)^\dagger\,|A^+(p, x_I^0)\rangle. \end{aligned} \tag{14}$$

The intermediate vacuum contribution is written as

$$\langle N_\rho(x^0); A(x_I^0)\rangle_{con}^{vac} = \left[i\frac{G_F}{\sqrt{2}}f_A\right]^2 \int_{x_I^0}^{x^0} dz^0 \int_{x_I^0}^{x^0} dy^0 \frac{1}{2E_A(p)V}\sum_{\mathbf{q}}\sum_{\mathbf{k}}\delta(\mathbf{k}+\mathbf{q},\mathbf{p})$$
$$\times \sum_{j,i,\sigma} Z_{\sigma j}^{1/2} Z_{\rho j}^{1/2*} Z_{\rho i}^{1/2} Z_{\sigma i}^{1/2*} \sum_{s,r} \bar{v}_{l\sigma}(qs)\not{p}(1+\gamma_5)u_j(kr)\rho_{ij}(k)\cdot \bar{u}_i(kr)\not{p}(1+\gamma_5)v_{l\sigma}(qs)$$
$$\times e^{i(\omega_j-\omega_i)x^0} e^{i(-\omega_j-E_\sigma+E_A)z^0} e^{i(\omega_i+E_\sigma-E_A)y^0}; \tag{15}$$

here, f_A is the decay constant of A^+ , defined by

$$\langle 0|\, j_a^{had}(x)^+ \left|A^+(p,x_I^0)\right\rangle = \frac{1}{\sqrt{2E_A(p)V}} i p_a f_A e^{i(px)} \exp(iE_A(p)x_I^0); \tag{16}$$

$v_{l\sigma}(qs)$ is the momentum-helicity eigenfunction for $\bar{l}_\sigma$, satisfying $(-i\not{q}+m_\sigma)v_{l\sigma}(qs)=0$, $q^0=E_\sigma(q)=\sqrt{\mathbf{q}^2+m_\sigma^2}$; $u_j^\dagger(kr)u_i(ks) = \rho_{ji}(k)\delta_{rs}$, $\rho_{ji}(k) = \cos[(\chi_j-\chi_i)/2]$ with $\cot\chi_j = |\mathbf{k}|/m_j$.[4, 5]. Some simplification of (15) is obtained if we multiply by hand to RHS of (15) a damping factor due to the decay width $\Gamma_A(p)$ of A-particle, i.e, $\exp[-\Gamma_A(p)(Z^0+Y^0)/2]$, $Z^0=z^0-x_I^0$, $Y^0=y^0-x_I^0$. Performing the Z^0- and Y^0-integrations under the condition $t\Gamma_A(p)\gg 1$, $t\equiv x^0-x_I^0$, we obtain

$$\langle N_\rho(x^0); A^+(x_I^0)\text{ with }\Gamma_A\rangle_{con}^{vac} = \left[i\frac{G_F}{\sqrt{2}}f_A\right]^2 \frac{1}{2E_A(p)V}\sum_{j,i,\sigma}\frac{1}{(2\pi)^3}\int d^3k$$
$$\times Z_{\sigma j}^{1/2} Z_{\rho j}^{1/2*} Z_{\rho i}^{1/2} Z_{\sigma i}^{1/2*}\rho_{ji}(k)\left[b(\sigma,q,p)_{\mathbf{d}}\;\hat{\mathbf{k}}\;\rho_{ji}^{(+)}(k) + ib(\sigma,q,p)_4\rho_{ji}(k)\right]$$
$$\times e^{i(\omega_j-\omega_i)t} I_j(k,q)I_i(k,q)^*\Big|_{\mathbf{q}=\mathbf{p}-\mathbf{k}}; \tag{17}$$

here, $b(\sigma,q,p)_d \equiv \sum_{a,b} b(\sigma,q)_d^{ab}p_ap_b$, $b(\sigma,q)_d^{ab} = -i\sum_s \bar{v}_{l\sigma}(qs)\gamma^a\gamma_d\gamma^b(1+\gamma_5)v_{l\sigma}(qs)$; $\rho_{ji}^{(+)}(k) = \cos[(\chi_j+\chi_i)/2]$; $I_j(k,q)$ comes from the Z^0-integration, concretely written as

$$I_j(k,q) = i/(\omega_j(k)+E_\sigma(q)-E_A(p)-i\Gamma_A(p)/2)\Big|_{\mathbf{p}=\mathbf{k}+\mathbf{q}}. \tag{18}$$

Owing to the **k**-integration, the oscillation behavior of (17) is different from that of the well-known standard formula [6, 7] as well as of the expectation value of the flavor neutrino charge with respect to a flavor neutrino state[8]. Detailed analyses in the plane- wave and the wave-packet $\left|A(x_I^0)\right\rangle$ cases will be done in a subsequent paper. (Some considerations on the latter case have been done in Ref. [9].)

3 Decay probabilities of neutrino source particles

We examine what results are obtained from $\langle N_\rho(x^0); A^+(p,x_I^0)\rangle$ for sufficiently large $T=x^0-x_I^0$ in the sense of $(x^0,x_I^0)\to(\infty,-\infty)$; from (15)

$$\langle N_\rho(x^0); A^+(x_I^0)\rangle_{con}^{vac}/T \longrightarrow \sum_{j,\sigma}|Z_{\rho j}^{1/2}|^2|Z_{\sigma j}^{1/2}|^2\frac{1}{(2\pi)^6}\int d^3k\int d^3q\frac{(2\pi)^4\delta^4(k+q-p)}{2E_A(p)}\left[\frac{G_F}{\sqrt{2}}f_A\right]^2$$
$$\times\sum_{s,r} i\bar{v}_{l\sigma}(qs)\not{p}(1+\gamma_5)u_j(kr)\cdot i\bar{u}_j(kr)\not{p}(1+\gamma_5)v_{l\sigma}(qs)$$
$$= {\sum_{j,\sigma}}' |Z_{\rho j}^{1/2}|^2|Z_{\sigma j}^{1/2}|^2 P(A^+(p)\to\bar{l}_\sigma\nu_j); \tag{19}$$

here, $P(A^+(p)\to\bar{l}_\sigma\nu_j) = P(A^+(\mathbf{p}=0)\to\bar{l}_\sigma\nu_j)\cdot(m_A/E_A(p))$; the sum $\sum_{j,\sigma}'$ is performed over j's and σ's which are allowed under 4-momentum conservation; the explicit form of the decay probability

$P(A^+(\mathbf{p}=0) \to \bar{l}_\sigma \nu_j)$ calculated in the lowest order of the weak interaction (7) is found e.g. in Refs. [5] and [10].

By taking into account various intermediate states as shown in (14), we can generalize (19) to each weak semileptonic decay, $A^+ \to \bar{l}_\sigma \nu_j +$ hadron(s) X; then, by summing over all modes, we obtain in order

$$\begin{aligned}
&\left[\langle N_\rho(x^0); A(x^0_I)\rangle_{con}/T\right]_{T\to\infty} (\equiv \langle N_\rho; A^+(p), \infty\rangle) \\
&= {\sum_{j,\sigma}}' |Z^{1/2}_{\rho j}|^2 |Z^{1/2}_{\sigma j}|^2 \left[P(A^+(p) \to \bar{l}_\sigma \nu_j) + \sum_{\text{modes}} P(A^+(p) \to \bar{l}_\sigma \nu_j + X) \right], \qquad (20)
\end{aligned}$$

$$\sum_\rho \text{Eq.}(20) = {\sum_{j,\sigma}}' \left|Z^{1/2}_{\sigma j}\right|^2 \left[P(A^+(p) \to \bar{l}_\sigma \nu_j) + \sum_{\text{modes}} P(A^+(p) \to \bar{l}_\sigma \nu_j + X) \right]. \qquad (21)$$

Judging from the meaning of the expectation value now considering, LHS of (21) may be regarded as the total leptonic decay probability of A^+-particle accompanying a neutrino; in the case of π^+, its life-time $\tau(\pi^+(p))$ is given by

$$\tau(\pi^+(p))^{-1} \simeq {\sum_{j,\sigma}}' \left|Z^{1/2}_{\sigma j}\right|^2 P(\pi^+(p) \to \bar{l}_\sigma \nu_j) \simeq {\sum_j}' \left|Z^{1/2}_{\mu j}\right|^2 P(\pi^+(p) \to \mu^+ \nu_j). \qquad (22)$$

It seems worthwhile to note the relation in the case of the expectation value of the number of charged lepton l_ρ. We obtain

$$\begin{aligned}
&\left[\langle A^+(p, x^0_I)\right| N^H_{l\rho}(x^0) \left|A^+(p, x^0_I)\rangle_{con}/T\right]_{T\to\infty} = {\sum_j}' |Z^{1/2}_{\rho j}|^2 \\
&\times \left[P(A^+(p) \to \bar{l}_\rho \nu_j) + \sum_{\text{modes}} P(A^+(p) \to \bar{l}_\rho \nu_j + X) \right] \qquad (23)
\end{aligned}$$

for energetically allowed ρ's. We see (23) leads also to (22).

Lastly we give a remark on the relations derived by assuming the 3 flavor number and $P(\pi^+(p) \to \mu^+\nu_j)$'s to be almost independent of all three j's as well as by noting $m_\tau \simeq 1780$ MeV. Then, from (20) we obtain

$$\begin{aligned}
&\langle N_{\nu_e}; \pi^+(p), \infty\rangle \;:\; \langle N_{\nu_\mu}; \pi^+(p), \infty\rangle \;:\; \langle N_{\nu_\tau}; \pi^+(p), \infty\rangle \\
&\simeq \sum_j |Z^{1/2}_{ej}|^2 |Z^{1/2}_{\mu j}|^2 \;:\; \sum_j |Z^{1/2}_{\mu j}|^4 \;:\; \sum_j |Z^{1/2}_{\tau j}|^2 |Z^{1/2}_{\mu j}|^2. \qquad (24)
\end{aligned}$$

Similarly in the case of K^+, under the j-independence of $P\big(K^+ \to \bar{l}_\sigma \nu_j (+\text{ X})\big)$ for $\sigma = e, \mu$, we obtain

$$\begin{aligned}
&\langle N_{\nu_e}; K^+(p), \infty\rangle \;:\; \langle N_{\nu_\mu}; K^+(p), \infty\rangle \;:\; \langle N_{\nu_\tau}; K^+(p), \infty\rangle \\
&\simeq \sum_j |Z^{1/2}_{ej}|^2 R_j \;:\; \sum_j |Z^{1/2}_{\mu j}|^2 R_j \;:\; \sum_j |Z^{1/2}_{\tau j}|^2 R_j \qquad (25)
\end{aligned}$$

with $R_j \equiv |Z^{1/2}_{ej}|^2 r(K^+ \to e^+) + |Z^{1/2}_{\mu j}|^2 r(K^+ \to \mu^+)$; here, $r(K^+ \to \bar{l}_\sigma)$ means the branching ratio of K^+ decay mode with one $\bar{l}_\sigma$. If it is allowed for us to drop $r(K^+ \to e^+)$ owing to the experimental value $r(K^+ \to e^+)/r(K^+ \to \mu^+) \simeq 4.87/66.70 \simeq 1/13.7$, we see RHS of (25) is nearly equal to that of (24).

As to the neutron life time, we also obtain

$$\left[\frac{1}{2} \sum_r \langle N_\rho(x^0); n(p, x^0_I)\rangle_{con}/T\right]_{T\to\infty} = {\sum_j}' |Z^{1/2}_{\rho j}|^2 |Z^{1/2}_{ej}|^2 P(n(\mathbf{p}) \to p e^- \bar{\nu}_j). \qquad (26)$$

Assuming the independence of $P(n \to p\ e\ \bar{\nu}_j)$ on j's, we obtain

$$\begin{aligned}
&\langle N_{\nu_e}; n(p), \infty\rangle \;:\; \langle N_{\nu_\mu}; n(p), \infty\rangle \;:\; \langle N_{\nu_\tau}; n(p), \infty\rangle \\
&\simeq \sum_j |Z^{1/2}_{ej}|^4 \;:\; \sum_j |Z^{1/2}_{\mu j}|^2 |Z^{1/2}_{ej}|^2 \;:\; \sum_j |Z^{1/2}_{\tau j}|^2 |Z^{1/2}_{ej}|^2. \qquad (27)
\end{aligned}$$

In the standard theory of the neutrino oscillation in vacuum[6], the transition probability for $\nu_\sigma \to \nu_\rho$ is given by $P_{\nu_\sigma\to\nu_\rho}(t = x^0 - x_I^0) = |\sum_j U_{\sigma j} e^{i\omega_j(\mathbf{k})t} U_{\rho j}^*|^2, \quad U^\dagger U = I$. The time average of $P_{\nu_\sigma\to\nu_\rho}(t)$[6, 7] over a sufficiently long time interval $T \gg 4\pi k/|\Delta m_{ji}^2|$ for any pair $(j,i),\ j \neq i$ is given by

$$\langle P_{\nu_\sigma\to\nu_\rho}\rangle_T \equiv \int_0^T dt P_{\nu_\sigma\to\nu_\rho}(t)/T = \sum_j |U_{\rho j}|^2 |U_{\sigma j}|^2, \tag{28}$$

which leads to the same ratio relations as (24) and (27) if we set $U = Z^{1/2}$. Although this equality is taken in the standard theory,[6, 7] there is some problem for understanding this equality on the basis of the quantum field theory, as first stressed by Blasone and Vitiello; a related remark is to be given below [4, 5, 11]. From our viewpoint, the relation (28) should be understood as the relations (24) and (27) which have a field theoretical basis.

4 Final remarks and conclusion

Through the above considerations, we can assert the following two points: Without recourse to one-neutrino flavor states, (i) a kind of the neutrino oscillation formula is derived from the main part of $\langle A(x_I^0)|\, N_\rho^H(x^0)\, |A(x_I^0)\rangle_{con}$ with finite $T = x^0 - x_I^0$, and (ii) the decay probabilities of hadronic A-particle accompanying a neutrino are obtained from $\langle N_\rho(x^0); A(x_I^0)\rangle_{con}/T$ for $T \to \infty$.

Concerning the way how to derive the theoretical expression of the decay probability of e.g. π^+, Giunti [13] has proposed a simplified version of the interacting wave-packet model,[14] where each one-neutrino flavor state is constructed by taking into account the related weak interaction. In this version, the amplitude for a sufficiently large T

$$\mathcal{A}_{\rho j}(k)_T \equiv \langle \nu_j(k)\bar{l}_\rho| - i\int_{-T/2}^{T/2} d^4y H_{int}(y)\, |\pi^+\rangle, \tag{29}$$

is defined. (For simplicity we omit the helicity.) Corresponding to the relevant weak interaction, the one flavor neutrino state is defined as

$$|''\nu_\rho(k)''\rangle_T \equiv e_\rho(k)\sum_j \mathcal{A}_{\rho j}(k)_T\, |\nu_j(k)\rangle \quad \text{with} \quad e_\rho(k) = \left[\sum_j |\mathcal{A}_{\rho j}(k)_T|^2\right]^{-1/2}. \tag{30}$$

We have $\langle ''\nu_\rho(k')''|''\nu_\rho(k)''\rangle = \delta(\mathbf{k}',\mathbf{k})$ for $\langle \nu_j(k')|\nu_i(k)\rangle = \delta_{ji}\delta(\mathbf{k}',\mathbf{k})$. Then, the transition amplitude for $\pi^+ \to'' \nu_\rho'' l_\rho$ is

$$\langle ''\nu_\rho(k)''\bar{l}_\rho| - i\int_{-T/2}^{T/2} d^4y H_{int}(y)\, |\pi^+\rangle = e_\rho(k)\sum_j \mathcal{A}_{\rho j}^*(K)_T \mathcal{A}_{\rho j}(k)_T = \frac{1}{e_\rho(k)}; \tag{31}$$

thus the probability of $\pi^+ \to'' \nu_\rho'' \bar{l}_\rho$ is given by

$$P(\pi^+ \to'' \nu_\rho'' \bar{l}_\rho) = \sum_{\text{phase vol}} \frac{1}{T e_\rho(k)^2} = \sum_j |Z_{\rho j}^{1/2}|^2 P(\pi^+ \to \bar{l}_\rho \nu_j), \tag{32}$$

which leads, after summing over allowed ρ's, to the relation corresponding to (22). As to the non-orthogonality problem, $\langle ''\nu_\rho(k')''|''\nu_\sigma(k)''\rangle \neq \delta_{\rho\sigma}\delta(\mathbf{k}',\mathbf{k})$, it should be noted that the orthogonality holds approximately if the dependence of $a_{\rho j}(k)_T \equiv (Z_{\rho j}^{1/2^*})^{-1}\mathcal{A}_{\rho j}(k)_T$ on j is negligible.

In this connection, the following remark on the state of one flavor neutrino (with 3-momentum and helicity $(\mathbf{k}, r)$) seems meaningful. We define

$$|\Psi_{\nu\rho}(\mathbf{k}\ r; x^0)^d\rangle \equiv \frac{1}{\sqrt{V}}\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} (\nu_\rho^\dagger(x)P(r))^d\, |0\rangle, \tag{33}$$

where $P(r)$ is the helicity projection matrix[3]-[5]; "d" specifies the 4-spinor component. As easily seen, the normalization is $\sum_d \left\langle \Psi_{\nu\sigma}(\mathbf{k}\ r; x^0)^d \middle| \Psi_{\nu\rho}(\mathbf{k}\ s; x^0)^d \right\rangle = \delta_{\sigma\rho}\delta(\mathbf{k}, \mathbf{k}')\delta_{rs}$, and the absolute square of the transition matrix element of $A^+(\mathbf{p}, x_I^0) \to \bar{l}_\sigma(\mathbf{q}\ s; x^0) + \nu_\rho(\mathbf{k}\ r; x^0)$, after summing the relevant quantities, leads to R.H.S. of (15). As easily seen, we obtain the standard neutrino oscillation formula[6, 7] from the absolute square of amplitude $\sum_d \left\langle \Psi_{\nu\sigma}(\mathbf{k}\ r; x_I^0)^d \middle| \Psi_{\nu\rho}(\mathbf{k}\ r; x^0)^d \right\rangle$ with $Z^{1/2} = U$. It is necessary for us make clear what differences in the oscillation behavior exist among the present expectation-value approach, the standard formula and the approaches of intermediate virtual neutrinos.[13, 14]

References

[1] H.Umezawa, Soryushiron (Misuzu Shyobo, Tokyo,1953), ch.13; Quantum Field Theory (North- Holland Pub.Co., Amsterdam, 1956), ch.13.

[2] C.N.Yang and D.Feldman, Phys.Rev. 79 (1950) 972; S.S.Schweber, Nuovo Cim. 11 (1955) 397

[3] E.Corinaldesi, Nucl.Phys. 7 (1958) 305

[4] K.Fujii, C.Habe and T.Yabuki, Phys.Rev. D59 (1999) 113003; 60 (1999) 099903(E).

[5] K.Fujii, C.Habe and T.Yabuki, Phys.Rev. D64 (2001) 013011

[6] S.M.Bilenky and B.Pontecorvo, Phys.Lett. B61(1976) 248; Phys.Rept. 41 (1978) 225; S.M. Bilenky, Neutrino masses, mixing and oscillations, lectures given at 1999 European School of High Energy Physics, Cast Papiernicka, Slovakia, Aug.22-Sep.4, 1992.

[7] E.g, M.Fukugita and T.Yanagida, Physics and Astrophysics of Neutrino, edited by M. Fukugita and A.Suzuki (Springer-Verlag, Tokyo, 1994), p.1.

[8] M.Blasone, P.A.Henning and G.Vitiello, Phys.Lett. B451 (1999) 140; M.Blasone, P.Jizba and G.Vitiello, Phys.Lett. B517 (2001) 471; M.Blasone, P.P.Pacheco and H.W.C.Tseung, Phys. Rev. D67 (2003) 073011

[9] K.Fujii and T.Shimomura, hep-ph/0402274.

[10] R.E.Shrock, Phys.Lett. B96 (1980) 159; Phys.Rev. D24 (1981) 1232; ibid. 24 (1981) 1275

[11] M.Blasone and G.Vitiello, Annals of Phys. 244 (1995) 283; 245 (1995) 363(E); Phys.Rev. D60 (1999) 111302

[12] C.Giunti, private communication.

[13] C.Giunti, C.W.Kim and J.A.Lee, Phys.Rev. D48 (1993) 4310; C.Giunti, C.W.Kim and U.W.Lee, Phys. Lett. B241 (1998) 237; C.Y.Cardall, Phys.Rev. D61 (2000) 073006; M.Beuthe, Phys.Rev. D66 (2002) 013003; Phys.Rep. 375 (2003) 105; C.Giunti, Theory of neutrino oscillations, lecture given in Sapporo Winter School 2004, Jan.8-12, 2004.

[14] T.Yabuki and K.Ishikawa, Prog.Theor.Phys. 108 (2002) 347; A.Asahara, K.Ishikawa, T.Shimomura and T.Yabuki, hep-ph/0406141.