On the numerical technique of Casimir energy calculation

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Abstract

A non-subtractive recipe of Casimir energy renormalization efficient in the presence of logarithmically divergent terms is proposed. It is demonstrated that it can be applied even in such cases, when energy levels can be obtained only numerically whereas neither their asymptotical behavior, nor the analytical form of the corresponding spectral equation can be studied. The results of numerical calculations performed with this method are compared to those obtained by means of explicit subtraction of divergent terms from energy.

1 Introduction

Ever since Casimir [1] has obtained corrections to the energy of a macroscopic system due to vacuum fluctuations of quantized electromagnetic field in 1948 this effect has been intensively studied both from theoretical and experimental points of view. Nevertheless the calculation of Casimir energy except for the most simple problems involving free fields inside cavities with flat boundaries remains quite non-trivial yet. To realize this, one can recall a great number of papers devoted to a free field confined in the interior of a sphere [2]–[11]. Despite the fact that in this case spectral equations can be written out explicitly, the first analytical results for massive scalar field have been obtained only in [12]. The analogous problem for fermions has been solved in [13].

It should be stressed that the knowledge of analytical form of spectral equation has been crucial for the employment of the method proposed in [12] since it makes possible the transition from the sums containing the unknown energy levels to the integrals with the explicit integrands [14]. The main goal of this paper is to propose a method which can be applied to numerical calculation of Casimir energy in situations when these requirements are not met. Another goal is to demonstrate that it's possible to perform all the necessary calculations without employment of a rather standard trick [7], [15], which lets one overcome problems arising due to the presence of logarithmic divergency in Casimir energy of free massless fields inside spherical shells. Note that logarithmic divergency appears as a consequence of a curved surface bounding the shell and makes the energy renormalization ambiguous. The main idea of the mentioned trick is to consider the "inner" and "extra" problems together since their logarithmic divergencies cancel each other.

The ambiguity of Casimir energy renormalization in the presence of logarithmic divergency is quite obvious. Indeed, in case of massless fields the energy of the system can be characterized by a single dimensional parameter L which is the linear size of the system. The regularization parameter α can be also chosen to have a dimension of length. In the absence of logarithmic divergency the "minimal subtraction" of singular terms is not only natural but also well grounded. Indeed, any divergent term in the expansion of regularized energy, which is proportional to α^{-s} ($s > 0$) is inevitably proportional to L^{s-1}, i.e. to the non-negative power of L. This makes it possible to normalize the final result at $L = \infty$, where Casimir energy should become zero, and subtract all singular terms at the same time. After such subtraction the only remaining term in the limit $\alpha \to 0$, which reads $c\alpha^0/L$, provides the final result.

In the presence of logarithmic divergency the subtraction becomes ambiguous, since in order to renormalize the term $c\alpha^0 \log(\alpha/L)/L$, one should subtract $c\alpha^0 \log(d\alpha/L)/L$, where d is an arbitrary constant, which cannot be determined from the normalizing condition at $L \to \infty$.

2 Massive scalar field in 1D

To illustrate the main idea of technique under investigation let's study Casimir energy with the logarithmic divergency in the most trivial case, i.e. Casimir energy of massive scalar field on an interval of length L

with Dirichlet boundary conditions at the ends of the interval. It reads:

$$\mathcal{E}_{cas} = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n = \frac{1}{2}\sum_{n=1}^{\infty}\sqrt{(\pi n/L)^2 + m^2} \tag{1}$$

First of all, the renormalization of Casimir energy is performed without intermediate subtraction of Minkowski vacuum contribution prescribed by "standard" approaches. The regularization of (1) requires the introduction of parameter α having a dimension of length, which stands in the argument of the cut-off function $F(\alpha\omega_n)$:

$$\mathcal{E}_{cas}^{(r)} = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n F(\alpha\omega_n) \tag{2}$$

Trivial considerations based on dimensional analysis lead to the following expression for the regularized Casimir energy

$$\mathcal{E}_{cas}^{(r)} \simeq c_{-2}\frac{L}{\alpha^2} + c_{-1}\frac{L^0}{\alpha^1} + c_0\frac{1}{L} + c_\lambda m^2 L\log(\alpha/L) + \cdots \tag{3}$$

It can be easily verified that for various cut-off functions such as $F(x) = \exp(-x)$, $F(x) = \exp(-x^2)$, $F(x) = \exp(-x^3)$, $\ldots$, $F(x) = \exp(-x^6)$, $F(x) = \exp(-2\cosh(x)+2)$, $\ldots$, identical c_λ are obtained, while c_0 are different. The identity of c_λ for different $F(x)$ can be demonstrated by means of the following estimation for the sum giving rise to the logarithmic divergency:

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2}\frac{m^2}{\omega_n}F(\alpha\omega_n) \sim \frac{m^2 L}{4\pi}\log N \sim \frac{m^2 L}{4\pi}\log(\Delta x L/\alpha) =$$

$$\frac{m^2 L}{4\pi}\log(L/\alpha) + \frac{m^2 L}{4\pi}\log(\Delta x)\ , \tag{4}$$

where $N \sim \Delta x L/\alpha$ and Δx is a cut-off interval of $F(x)$.

Since any subtraction in the presence of logarithmic divergency is ambiguous this procedure should be excluded from consideration along with the logarithmic divergency itself. To achieve that one should calculate $\partial_L^2\mathcal{E}_{cas}^{(r)}$:

$$\partial_L^2\mathcal{E}_{cas}^{(r)} \simeq c_0\frac{2}{L^3} + c_\lambda\frac{m^2}{L} + \cdots \tag{5}$$

The obtained expression is regular in the limit $\alpha \to 0$, so no subtraction is required. The knowledge of the function $\partial_L^2\mathcal{E}_{cas}^{(r)}$, lets one reconstruct the required $\mathcal{E}_{cas}^{(r)}$ unambiguously, since the initial conditions at $L \to \infty$ are well-known: both $\mathcal{E}_{cas}^{(r)}$ and $\partial_L\mathcal{E}_{cas}^{(r)}$ should become zero in this limit. Note that while (5) doesn't describe the asymptotical behavior of $\partial_L^2\mathcal{E}_{cas}^{(r)}$ at $L \to \infty$, it demonstrates the disappearance of all singular terms in this quantity. Moreover the following integral approximation shows that $\partial_L^2\mathcal{E}_{cas}^{(r)}$ does really vanish for $L \to \infty$:

$$\mathcal{E}_{cas}^{(r)} = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n F(\alpha\omega_n) = \frac{1}{4}\sum_{n=-\infty}^{\infty}\omega_n F(\alpha\omega_n) - \frac{m}{4}F(\alpha m) \approx$$

$$\approx \frac{1}{2}\int_{-\infty}^{\infty}dx(L/\pi)\sqrt{x^2+m^2}F(\alpha\sqrt{x^2+m^2}) - \frac{m}{4}F(\alpha m) \tag{6}$$

3 Method of calculation in general case

The proposed method can be generalized in such a way that it doesn't require the analytical expression for energy levels for its application. Suppose one has a set of energy levels of some spectrum ω_n and the corresponding regularized Casimir energy contains the logarithmically divergent term. First of all it turns out to be possible to modify the initial expression for the Casimir energy by introduction of some parameter μ in such a way that

$$\frac{1}{2}\sum_{n=1}^{\infty}\sqrt{\omega_n^2-\mu^2}F(\alpha\sqrt{\omega_n^2-\mu^2}) \tag{7}$$

no longer contains the logarithmic divergency. For the massive scalar field on an interval μ obviously equals to the mass of the field. In less trivial three-dimensional cases with spherical symmetry μ is some parameter having a dimension of mass which characterizes the total coefficient by the logarithmic divergency with all values of angular momentum taken into account.

The next step is to introduce an "additional mass" of the field $\mathcal{M}$ and study the behavior of Casimir energy in the range from $\mathcal{M}=0$ to $\mathcal{M}=\infty$. In fact it's convenient to deal with another parameter M, which is related to $\mathcal{M}$ by means of $M^2 \equiv \mathcal{M}^2+\mu^2$, and study the "modified" Casimir energy

$$\mathcal{E}_{cas}(M)=\frac{1}{2}\sum_{n=1}^{\infty}\sqrt{\omega_n^2+M^2-\mu^2} \tag{8}$$

as a function of M in the range from $M=\infty$ to $M=\mu$. To carry this out, one should calculate numerically the following quantity in the specified range of M:

$$\partial_M^2\left(\mathcal{E}_{cas}^{(r)}(M)/M\right)=$$

$$=\partial_M^2\left[\frac{1}{2}\sum_{n=1}^{\infty}\sqrt{\frac{\omega_n^2-\mu^2}{M^2}+1}\;\; F\left(\alpha\sqrt{\frac{\omega_n^2-\mu^2}{M^2}+1}\right)\right] \tag{9}$$

Note that in contrast to our previous considerations the dimensionless quantity

$$\frac{1}{M}\sqrt{\omega_n^2+M^2-\mu^2}=\sqrt{\frac{\omega_n^2-\mu^2}{M^2}+1}$$

has been substituted to the argument of $F(x)$, so that α should be also taken dimensionless.

An alternative interpretation of (9) follows from the observation that M acts just as an effective length L in the expression for Casimir energy. Indeed, (9) can be obtained from the initial expression as a result of the following sequence of transformations of the spectrum. At the first step ω_n is transformed to $\omega_n'=\sqrt{\omega_n^2-\mu^2}/\mu$, which is the dimensionless form of the spectrum with the subtracted effective mass. After that the scale transformation of the system $x\to x(M/\mu)$ and $\omega_n'\to\omega_n''=\omega_n'/(M/\mu)$ is performed. In the end the unit mass is "added" to the obtained spectrum:

$$\omega_n''\to\sqrt{(\omega_n')^2+1}=\sqrt{\frac{\omega_n^2-\mu^2}{M^2}+1} \tag{10}$$

The limit $M\to\infty$ obviously corresponds to the infinite size of the system, while for $M=\mu$ one obtains the initial spectrum divided by μ.

It's easy to see that all divergent terms in (9) vanish. In the limit $n\to\infty$ one can make use of the following expansion

$$\sqrt{\frac{\omega_n^2-\mu^2}{M^2}+1}\approx\frac{\sqrt{\omega_n^2-\mu^2}}{M}+\frac{M}{2\sqrt{\omega_n^2-\mu^2}}+\cdots \tag{11}$$

The first term in this expansion gives rise to the sum which is free of logarithmic divergency owing to the definition of μ. Other divergencies are proportional to $(M/\alpha)^2/M$ and $(M/\alpha)^1/M$ and vanish when the second-order derivative is taken. The second term leads to the logarithmic divergency with the coefficient by it proportional to M^1, which also vanishes while taking the second-order derivative.

As a result one has the expression (9) regular in the limit $\alpha\to 0$ and the natural normalizing condition $\mathcal{E}_{cas}(M\to\infty)=0$. The latter can be understood from two different points of view. On one hand the quantized field with the infinitely large mass should definitely have zero Casimir energy. On the other hand the Casimir energy in the limit of the infinite size of the system should become zero. Whichever interpretation is chosen, these two points let one reconstruct the required $\mathcal{E}_{cas}(M=\mu)$ corresponding to the non-modified spectrum.

Note, that principally one could consider $\mathcal{E}^{(r)}_{cas}(M)$ instead of $\mathcal{E}^{(r)}_{cas}(M)/M$. However that would increase the order of derivative required to exclude all divergent terms what is undesirable from the point of view of real-time computations.

The proposed method turns out to be efficient not only in the most trivial one-dimensional cases, but also in more realistic three-dimensional ones. However to employ it in three-dimensional case one should inevitably calculate the fourth-order derivative of the Casimir energy (9) since the main singular term, which is proportional to the volume of the system, reads $c_{-4}L^3/\alpha^4$. It should be also noted that in this case the summations become more lengthy and sophisticated. Typically the final result turns out to be about 40 orders lower than the values of partial sums obtained in the process of its calculation. As a consequence, extra floating-point precision is required.

4 Numerical results

For scalar field on an interval $[0; L]$ with $L = 1$ the spectrum reads

$$\omega_n = \sqrt{\frac{(\pi n)^2}{L^2} + m^2} \tag{12}$$

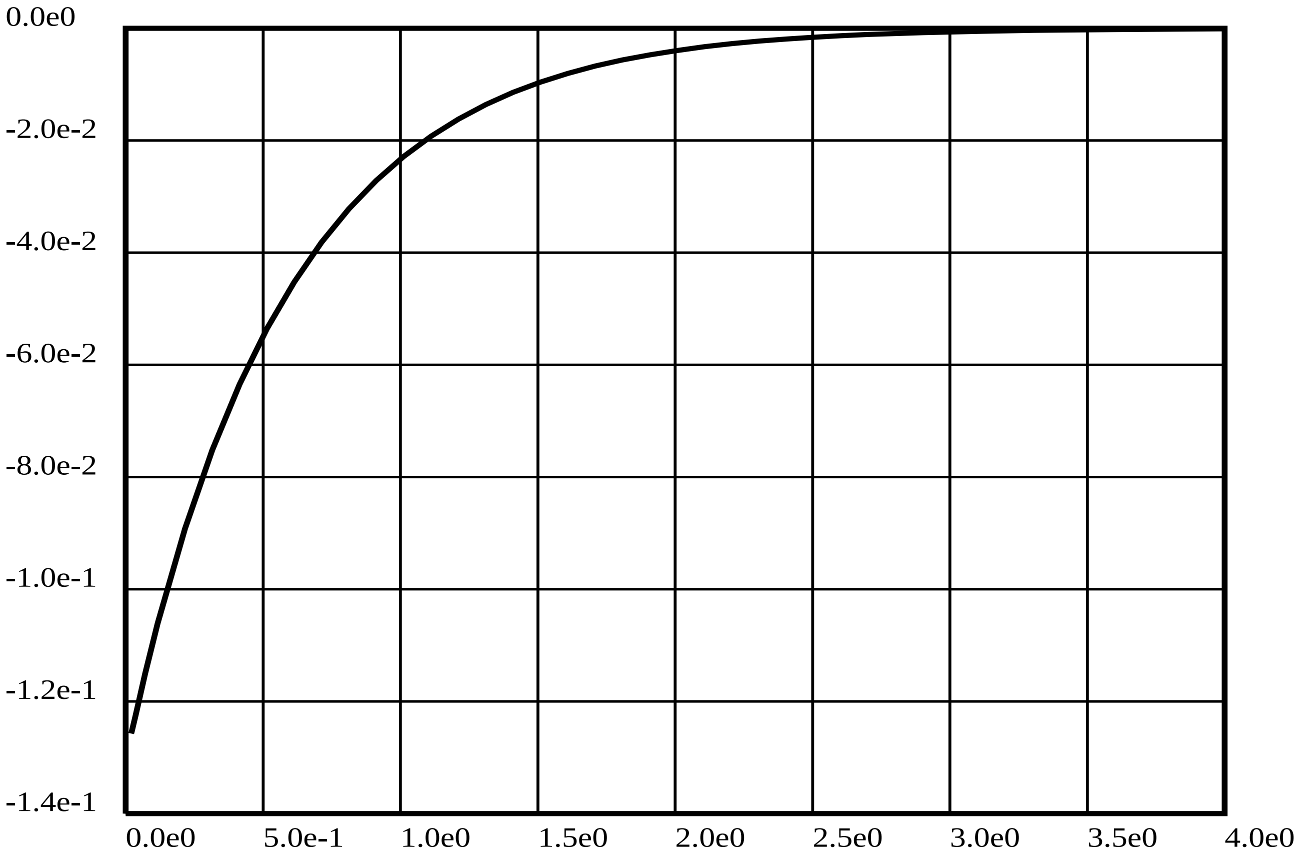

Fig. 1: The Casimir energy of the massive scalar field on a unit interval as a function of the mass of the field.

The result of straightforward application of the proposed technique with various cut-off functions such as $F(x) = \exp(-x)$, $F(x) = \exp(-x^2)$, $F(x) = \exp(-x^3)$, ..., $F(x) = \exp(-x^6)$, $F(x) = \exp(-2\cosh(x) + 2)$ is presented on Fig. 1. It has been shown that for each of these functions the same result is obtained and what's more the precision of coincidence depends only on the number of energy levels taken into account and the number of right digits used in the realization of floating-point arithmetics as well.

As to dependence of the Casimir energy on the mass of the field some important aspects should be stressed. First of all in the limit $m \to 0$ a well-known result for the massless scalar field is obtained. In the range of large values of m Casimir energy decreases exponentially as e^{-2mL} what could be expected from qualitative considerations. To summarize, the results obtained in this trivial case are in full agreement with the previously known results, obtained with the traditional technique, based on an explicit subtraction of divergent terms.

To demonstrate how this approach can be employed in less trivial cases it makes sense to consider the massless scalar field inside of a spherical shell of radius $R = 1$. In this case the same set of cut-off functions has been exploited. The values of an effective mass $\mu = 0.1377$ obtained with each of these functions coincide up to the first four digits. Consequently the precision of the obtained $E_{cas} = 3.790 \cdot 10^{-3}$ has the same order, what corresponds to about 200 s-levels taken into account while performing the calculations. In fact, the number of energy levels taken into account is directly affected by the range which the regularization parameter α used in the calculations belongs to. Therefore one can control precision of the final result simply changing the range of employed values of the regularization parameter.

Note that in the framework of this approach not only Casimir energy of the massless scalar field (corresponding to $\mathcal{M} = 0$) has been obtained but also Casimir energy for all possible mass values in the range from zero to the "effective" infinity. The dependence of the Casimir energy of the scalar field inside the sphere on the mass of the field is presented on Fig. 2.

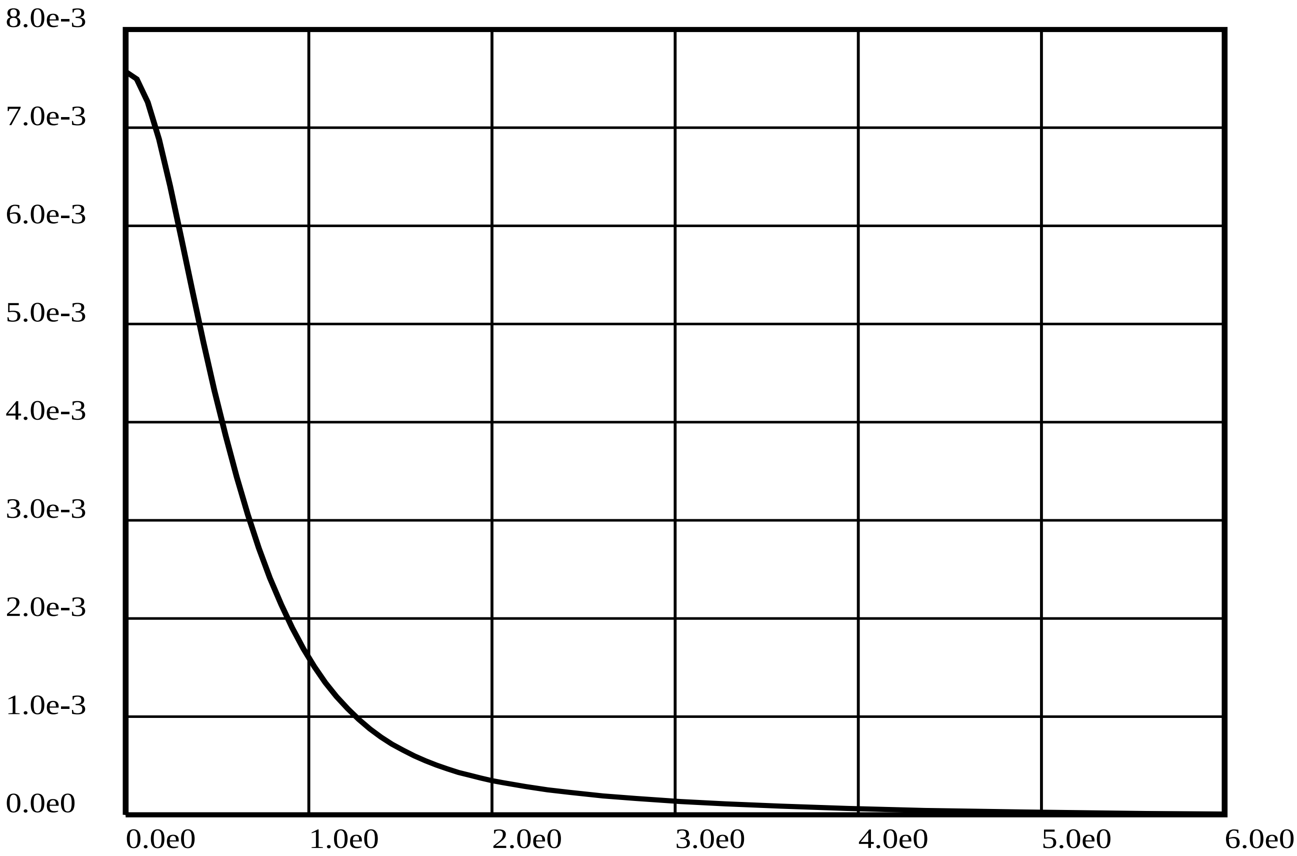

Fig. 2: The Casimir energy of the scalar field inside of a spherical shell of radius 1 obeying Dirichlet boundary conditions as a function of the mass of the field.

It should be stressed that on the contrary to the results obtained with the traditional subtractive technique in [12] our result doesn't contain logarithmical singularity at $\mathcal{M} = 0$ what seems more reasonable from the physical point of view. The most likely explanation of this discrepancy is that the subtractive procedure may very well contain some arbitrariness, even if one puts the proper normalizing condition. As a result some function which has regular behavior at $\mathcal{M} \to \infty$ (and thus doesn't violate the normalizing condition), but is singular at $\mathcal{M} \to 0$ may have been subtracted from the final result.

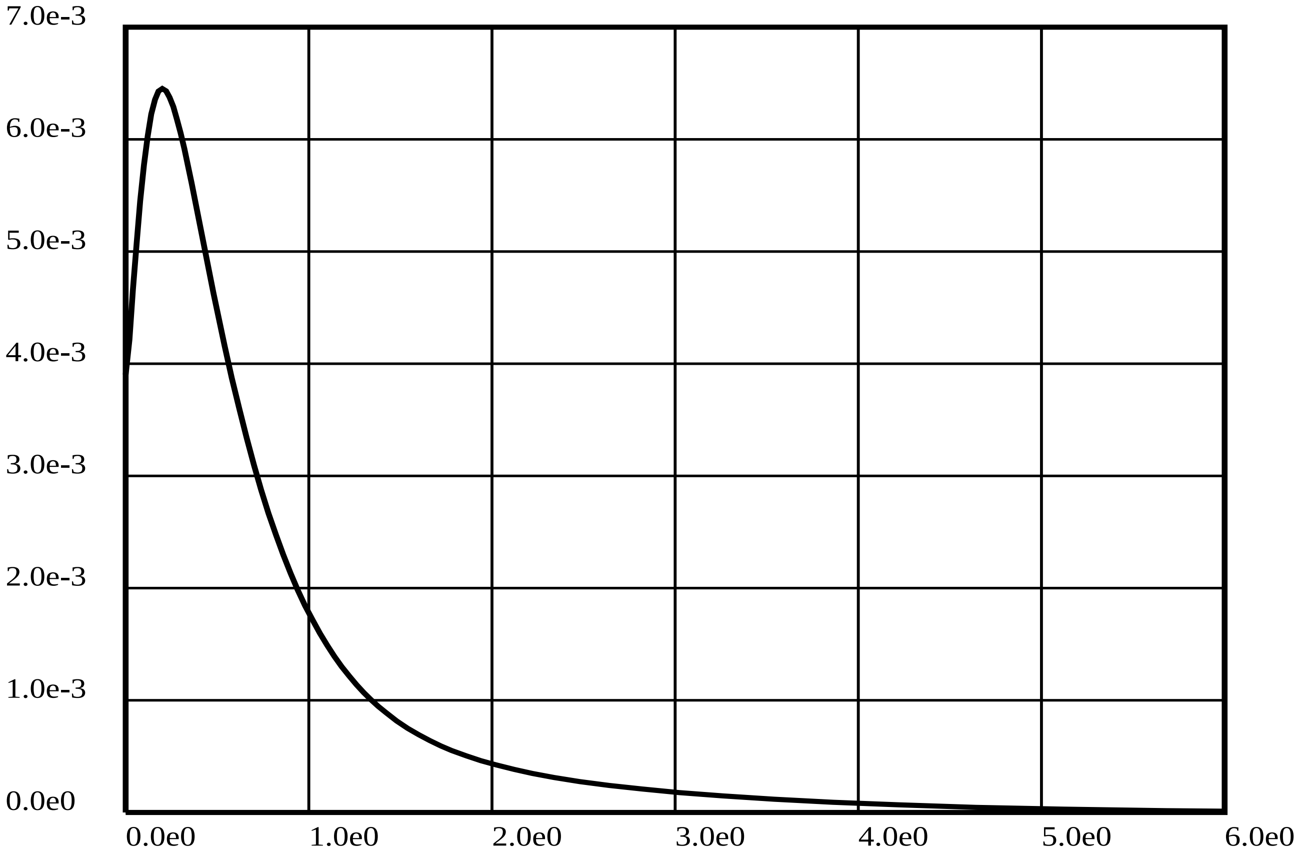

Fig. 3: The Casimir energy of a massive scalar field in the whole space with Dirichlet boundary condition on a sphere of radius 1 as a function of the mass of the field.

As has been pointed out in [12], [15] there is no argument at present which can remove this arbitrariness in case of a massless scalar field in the interior (or exterior) of a sphere. Therefore Casimir effect in the whole space with Dirichlet boundary conditions on the sphere is usually considered instead of interior alone (see the Introduction). It seems instructive to calculate the Casimir energy in this specific case employing our technique.

In fact there are two ways to proceed to take exterior into account. The first one deals with the continuous spectrum and requires that all the regularized sums be replaced with the appropriate integrals containing the energy levels density in their integrands. The second way lets one work with discrete spectrum all the time. To carry that out one should place the initial spherical shell into another sphere with the radius $R_{out} = kR_{in}$ where $k \geq 1$ and calculate the Casimir energy for the system bounded by the outer sphere with the boundary conditions on both of the spheres taken into account. For each finite k the spectrum remains discrete and the developed technique can be applied without any modification. The required result can be achieved in the limit $k \to \infty$. In practice it turns out that for $k \geq k_0$, where k_0 is finite and depends on the required precision only, the result doesn't depend on k. It turns out that in the case under investigation the 4-digit precision of the final result can be achieved with $k_0 \simeq 10$.

The final result of the calculations is presented on Fig. 3. Note that while the qualitative behavior of Casimir energy is the same as that obtained with methods employing explicit subtraction [15], there is no absolute coincidence. For example, for the massless field the result obtained with our method is $\mathcal{E}_{cas}(M=0) = 0.0039$ while direct subtraction leads to $\mathcal{E}_{cas}(M=0) = 0.0028$. Again, the remaining discrepancy can be related to the finite arbitrariness of the subtractive procedure.

5 Conclusion

To summarize, an efficient technique for numerical calculation of Casimir energy of quantized fields in the presence of logarithmical divergencies, owing to non-trivial boundary conditions, has been developed. The advantages of the proposed method are its ideological simplicity and universality which let one apply it to a wide range of problems in which numerical values for energy levels can be obtained with the sufficient precision. As has been demonstrated the results of its application to a number of problems appear to be quite reasonable, including the cases with curved boundaries. As to possible disadvantages they turn out to be mostly technical: the required floating-point precision to carry out all the necessary calculations in realistic cases turns out to be quite high (from three to four times higher than the one realized in the standard C double type).

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