The analytical approach to constructing effective nucleon-nucleon interaction operators at low and intermediate energies

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Abstract

A new manifestly Poincare-invariant approach to constructing effective local hadron-hadron interaction operators, based on the analytical theory of a S-matrix and methods of solving the inverse scattering problem, is developed. The suggested approach is applied to constructing the nucleon-nucleon effective potentials in $^1S_0$ - and $^3S_1$ - states.

1 Introduction

The force between two nucleons is one of the most fundamental problems in nuclear and particle physics. It is now widely believed that QCD is fundamental theory of strong interactions. On this basis all hadron-hadron interactions are completely determined by the underlying quark-gluon dynamics. However, due to the formidable mathematical problems raised by the non-perturbative character of QCD in the low- and intermediate-energy regime, we are still far from a quantitative understanding hadron-hadron interactions from this point of view. The path-integral method together with the idea of spontaneous violation of chiral invariance leads to chiral effective theory (ChET) of strong interactions. The ChET formulates the theory of hadron-hadron interactions in terms of meson and baryon (anti-baryon) degrees of freedom. The Lagrangian of ChET is highly nonlinear and very complicated. Therefore in practice the decomposition of this Lagrangian in power series on relative momentum of particles is used and in this approximation the theory refers to as chiral perturbation theory (ChPT). Unfortunately ChPTcan be applied to description of strong interactions only at very low energies.

On the other hand, in nuclear physics for a long time the meson conception of nuclear force is used. Now it is not quite clear, up to what distances the meson picture of nuclear force is valid and what degrees of freedom determine the basic properties of the short-range components of effective hadron-hadron interactions. The meson theories face with following basic difficulties: i) underbinding of a few-nucleon systems and ii) an explanation of short-range repulsion in NN-interaction. For example, in Bonn potential [1] the short-range repulsion is due to ω-meson exchange. It is required in this theory non-realistically large (≈ 20) value of a coupling constant $g^2_{\omega NN}/4\pi$, considerably surpassing the value following from experimental data [2] and also from the quark-model calculations. In the present work the new relativistic approach to constructing the nucleon-nucleon effective interaction operators, based on the analytical theory of S-matrix and the methods of inverse scattering problem, is suggested.

2 Definition of effective interaction operators and its constructing by methods of inverse quantum scattering problem

Requirements of unitarity and analyticity with inevitability lead to that the S-matrix should have the cuts on invariant masses of systems of two, three etc. particles in all possible channels of scattering. A principle of the maximal analyticity is meant, that the S-matrix, besides the poles corresponding to one-particle states, contains only those singularities which are necessary for performance of conditions of unitarity. The discontinuities of the partial-wave scattering amplitudes on dynamic cuts are determined by model-independent characteristics - renormalized coupling constants in three-line vertices and amplitudes of elementary subprocesses with on-mass-shell particles (Cutkosky cutting rules [3]). Therefore the structure at least the nearest to physical region cuts of the partial-wave scattering amplitudes can be determined in model-independent manner. For description of dynamics of the NN-interaction we shall use the relativistic three-dimensional equation of quasi-potential type, which can be obtained by reconstruction of initial Bethe-Salpeter equation [4]. Let's consider for simplicity one channel scattering of two particles with equal masses m. In momentum representation the three-dimensional equation for Feynman partial-wave scattering amplitude with the orbital angular momentum l has the form

$$A_l(k',k;s) = W_l(k',k;s) + \frac{2}{\pi}\int_0^\infty \frac{W_l(k',q;s)A_l(q,k;s)}{q^2-k^2-i\eta}\frac{q^2dq}{\sqrt{q^2+m^2}}, \tag{1}$$

where $s=k^2$ is a square of a three-dimensional relative momentum of colliding particles in the center-of-mass system. Redefining partial-wave scattering amplitudes and matrix elements of the interaction operator by means of relations

$$A_l(k',k;s) = (k'^2+m^2)^{1/4}T_l(k',k;s)(k^2+m^2)^{1/4}, \tag{2}$$

$$W_l(k',k;s) = (k'^2+m^2)^{1/4}V_l(k',k;s)(k^2+m^2)^{1/4}, \tag{3}$$

we obtain the equation of a Lippmann-Shwinger type

$$T_l(k',k;s) = V_l(k',k;s) + \frac{2}{\pi}\int_0^\infty \frac{V_l(k',q;s)T_l(q,k;s)}{q^2-k^2-i\eta}q^2dq. \tag{4}$$

We shall note, that dependence of the matrix elements $V_l(k',k;s)$ on spectral parameter s results in that the appropriate effective interaction operator has no property of a self-conjugating.

As follows from a principle of maximal analyticity, the on-mass-shell Feynman partial-wave scattering amplitude $A_l(k^2)=A_l(k,k;k^2)$ in a complex $s=k^2$-plane has i) the poles corresponding to bound states, ii) the unitary (right-hand) cut and iii) the dynamic (left-hand) cuts caused by exchange processes in t-and u-channels of scattering. Let's consider at first stage only the elastic scattering (we will take into account the effects of inelasticity in the final version of the developed method). The basic idea of the suggested approach is to define effective potential as such local operator in Lippmann-Shwinger partial-wave equation, which generates relativistic (Feynman) on-mass-shell scattering amplitude, having required discontinuities on dynamic cuts. Specified discontinuities of the partial-wave scattering amplitudes can be calculated by methods of the relativistic quantum field theory with the help of various dynamic approaches. In particular, for calculation of this discontinuities on the sites of dynamic cuts nearest to physical region the chiral perturbation theory can be used.

Connection between discontinuities of the amplitudes on dynamic cuts and effective local potentials in the given partial-wave states can be established with the help of Marchenko equation [5]

$$K_l(r,r') = B_l(r,r') + \int_r^\infty K_l(r,t)B_l(t,r')dt, \tag{5}$$

the kernel of which

$$B_l(r,r') = \frac{1}{2\pi}\int_{-\infty}^\infty h_l^{(+)}(kr)[S_l(k)-1]h_l^{(+)}(kr')dk + \sum_n \gamma_{nl}h_l^{(+)}(i\xi_{nl}r)h_l^{(+)}(i\xi_{nl}r') \tag{6}$$

(where $h_l^{(+)}(kr)$ are the Riccatti-Hankel functions) is determined by a partial-wave S-matrix $S_l(k)$ on the real axis in a complex plane of a relative momentum of colliding particles and its residues

$$\gamma_{nl} = -i\lim_{k\to i\xi_{nl}}[(k-i\xi_{nl})S_l(k)] \tag{7}$$

in the poles on the physical sheet, corresponding to the bound states. The matrix element of the local interaction operator in coordinate representation (potential) $V_l(r)$ is connected to the solution of the Marchenko equation by relation

$$V_l(r) = -2\frac{d}{dr}K_l(r,r). \tag{8}$$

Let's close a contour of integration in expression (6) in upper half-plane, bending around a cut along an imaginary axis from a point $k=i\mu/2$ up to $i\infty$ and the poles corresponding to the bound states. Here the value μ determines the position of the point of a dynamic cut nearest to physical region. For NN - scattering $\mu = m_\pi$, where m_π is mass of π-meson. We shall take into account also, that contributions from the poles, corresponding to bound states, in this integral in accuracy are canceled with the second term in the formula (6). As a result the kernel of the integral equation (5) is expressed through discontinuities of the partial-wave S-matrix along the specified dynamic cut

$$S_l(i\xi + \varepsilon) - S_l(i\xi - \varepsilon) = 2iD_l(\xi) \tag{9}$$

(where $D_l(\xi)$ is real function of a variable $\xi = -ik$) by equation

$$B_l(r, r') = -\frac{1}{\pi}\int_{\mu/2}^{\infty} h_l^{(+)}(ir\xi) D_l(\xi) h_l^{(+)}(ir'\xi). \tag{10}$$

The functions $D_l(\xi)$ are connected to discontinuities $\sigma_l(-s)$ of Feynman partial-wave scattering amplitudes $A_l(s)$ on a dynamic (left-hand) cut C_L in a complex $s = k^2$ plane by relation

$$D_l(-ik) = \frac{2ik\sigma_l(-k^2)}{\sqrt{m^2 + k^2}}, \tag{11}$$

where

$$\sigma_l(-s) = \frac{1}{2i}[A_l(s + i\varepsilon) - A_l(s - i\varepsilon)] \qquad \text{on} \quad C_L. \tag{12}$$

3 Comparison of the suggested approach with the one-boson-exchage model

As an example in the given work we apply the suggested approach to calculation of effective NN-potentials in ${}^1S_0$- and ${}^3S_1$-states. In this section on a simple example we shall carry out the comparative analysis of the suggested approach with the one-boson-exchange model. The one meson exchange mechanism generates a dynamic cut of the NN-partial-wave scattering amplitudes in a complex s-plane on the real axis at $s \le -\mu^2/4$, where μ is mass of the meson. We shall consider a case of S-wave scattering (an index $l = 0$ further we omit). Let's present the discontinuities of partial-wave scattering amplitudes $A_l(s)$ on this cut as

$$\sigma(\xi^2) = \frac{\pi}{4\xi^2} g(\xi^2)\sqrt{m^2 - \xi^2}\,\theta(\xi; \xi_1, \xi_2), \tag{13}$$

where $\theta(\xi; \xi_1, \xi_2) = \theta(\xi - \xi_1)\theta(\xi_2 - \xi)$, $\theta(x) = 1$ (0) at $x > 0$ ($x < 0$). The parameter $\xi_1 = \mu/2$ determines the point of a dynamic cut nearest to physical region and ξ_2 is cut-off parameter. As a example we consider the model, in which the function $g(\xi^2)$ is constant g_0 (it is possible to show, that this function is constant for meson-exchange mechanism with the zero-spin meson in a non-relativistic limit). The cut-off parameter ξ_2 in the formula (13) we shall put equal to μ (so the point $\xi = \mu$ coincides with the beginning of the two-meson cut). Within the framework of this model we shall obtain, using the Marchenko equation (5), analytical expression for potential up to the second order in a constant g_0. First of all, we shall present required potential in the form

$$V(r) = \int_\mu^\infty C(\alpha)\exp(-\alpha r)d\alpha, \tag{14}$$

which guaranties required position of a dynamic cut. Calculation of function $C(\alpha)$ results in the following expression: $C(\alpha) = \sum_{i=1}^{3} C_i(\alpha)$, where

$$C_1(\alpha) = g_0\,\theta(\alpha; \mu, 2\mu), \quad C_2(\alpha) = \frac{2g_0^2}{\alpha}\ln\left(\frac{\alpha - \mu}{\mu}\right)\theta(\alpha; 2\mu, 3\mu), \quad C_3(\alpha) = \frac{2g_0^2}{\alpha}\ln\left(\frac{2\mu}{\alpha - 2\mu}\right)\theta(\alpha; 3\mu, 4\mu).$$

With the help of these formulas it is possible to obtain obvious expression for potential in the second order with respect to the constant g_0

$$V(r) = g_0[\exp(-\mu r) - \exp(-2\mu r)]/r + g_0^2\ [\mathrm{Ei}(-\mu r) - \mathrm{Ei}(-2\mu r)]^2 + O(g_0^3), \tag{15}$$

where $\mathrm{Ei}(x)$ is known special function. We shall notice, that the terms of the third and higher power give in potential the contributions decreasing at $r \to \infty$ not slowly, than $\exp(-3\mu r)$. Asymptotic decomposition of the second term in Eq. (15) at $r \to \infty$ is given by

$$V^{(2)}(r) = \frac{g_0^2 \exp(-2\mu r)}{\mu^2 r^2}\left(1 - \frac{2}{\mu r} + \frac{5}{\mu^2 r^2} - \frac{16}{\mu^3 r^3} + ...\right) + O(\exp(-3\mu r)). \tag{16}$$

Thus, in the suggested approach the account of the discontinuity, corresponding to the one meson exchange mechanism, generates in potential contributions of the second and higher power in a constant g_0. The one-meson exchange potential contains only the linear in g_0 contribution.

Determination of the function $C(\alpha)$ in the Eq. (14) on the basis of the information about discontinuities of partial-wave scattering amplitudes on dynamic cuts can be realized also with the help of the integral equations suggested by A.Martin [6]. For S-wave scattering these equations have the form

$$g(\alpha^2/4) = C(\alpha) + \theta(\alpha - 2\mu)\int_{\mu}^{\alpha-\mu} C(\alpha - \beta)\rho(\alpha, \beta)d\beta, \tag{17}$$

$$\alpha(\alpha - \gamma)\rho(\gamma, \alpha) = C(\alpha) + \theta(\alpha - 2\mu)\int_{\mu}^{\alpha-\mu} C(\alpha - \beta)\rho(\gamma, \beta)d\beta. \tag{18}$$

It follows from Eqs. (20), (21) that i) at the fixed value of a variable α the function $C(\alpha)$ is determined by discontinuity $\sigma(\xi^2)$ only at $\xi^2 \le \alpha^2/4$ and ii) to find $C(\alpha)$ only finite number of iterations is needed.

4 Construction of effective NN-potentials in $^1S_0$- and $^3S_1$-states

Let's present discontinuities of S-wave NN-amplitudes in a states with full angular momentum J as

$$\sigma_J(\xi^2) = \frac{\pi m_N^2}{4\xi^2}\sum_{\lambda} f_\lambda^{(J)}(-\xi^2/m_N^2) h_\lambda^{(J)}(2\xi), \tag{19}$$

where we have entered an index λ designating one of the mesons: $\pi, \sigma, \rho, \omega, \eta$ and a_0; m_N is mass of nucleon; functions $h_\lambda^{(J)}(\alpha)$ set borders of regions in which discontinuities are distinct from zero and besides contain factors which cut-off the contributions of distant from physical region singularities. Calculation of the functions $f_\lambda^{(J)}(x)$ for NN-scattering in $^1S_0$- and $^3S_1$-states results in the following expressions:

$$f_\lambda^{(J)}(x) = \frac{g_{\lambda NN}^2 \delta_\lambda}{48\pi}\,\gamma_\lambda \tag{20}$$

are contributions of the π- and η- mesons, where $\gamma_\lambda = m_\lambda^2/m_N^2$, m_λ is mass of λ-meson, $\delta_\pi = -3$ and $\delta_\eta = 1$;

$$f_\lambda^{(0)}(x) = -\frac{g_{\lambda NN}^2 \delta_\lambda}{4\pi}\left(1 - \frac{\gamma_\lambda}{4}\right), \tag{21}$$

$$f_\lambda^{(1)}(x) = -\frac{g_{\lambda NN}^2 \delta_\lambda}{4\pi}\left\{1 - \frac{\gamma_\lambda}{12}\left(1 + \frac{\gamma_\lambda}{x}\right) + \frac{2\gamma_\lambda}{3x}\left(1 + \frac{\gamma_\lambda}{4x}\right)[1 - \sqrt{1+x}]\right\} \tag{22}$$

are contributions of the σ- and a_0- mesons, where $\delta_\sigma = 1$ and $\delta_{a_0} = -3$;

$$f_\lambda^{(0)}(x) = -\frac{g_{\lambda NN}^2 \delta_\lambda}{12\pi}\left\{1 + 2x - \frac{\kappa_\lambda \gamma_\lambda}{2}\left[1 + \kappa_\lambda\left(1 + x + \frac{\gamma_\lambda}{8}\right)\right]\right\}, \tag{23}$$

$$f_\lambda^{(1)}(x) = \frac{g_{\lambda NN}^2 \delta_\lambda}{4\pi}\left\{1 + 2\gamma_\lambda - \frac{\kappa_\lambda \gamma_\lambda}{2}\left(1 - \kappa_\lambda\gamma_\lambda - \frac{\kappa_\lambda}{3} + \frac{\kappa_\lambda^2}{8}\right) + x\left(2 + \frac{7\gamma_\lambda\kappa_\lambda^2}{24} - \frac{\kappa_\lambda^3}{8}\right) + \right.$$

$$\left. + \frac{\gamma_\lambda^2}{12x}\left(1 + \frac{\gamma_\lambda\kappa_\lambda^2}{4}\right) + \frac{2\gamma_\lambda}{3x}\left[1 - \frac{\kappa_\lambda\gamma_\lambda}{2}\left(1 - \kappa_\lambda\right) + \frac{\gamma_\lambda}{4x}(1 + \frac{\gamma_\lambda\kappa_\lambda^2}{4}\right)\right](1 - \sqrt{1+x}) + \frac{4\kappa_\lambda\gamma_\lambda}{3}\sqrt{1+x}\right\} \tag{24}$$

are contributions of the ρ - and ω - mesons, where $\delta_\rho = -3$, $\delta_\omega = 1$, $\kappa_\omega = 0$ and $\kappa_\rho = f_{\rho NN}/g_{\rho NN}$. We have adopted the same definitions of the meson-nucleon coupling constants as in Ref. [1].

Functions $h_\lambda^{(J)}(\alpha)$ are represented as

$$h_\lambda^{(J)}(\alpha) = \theta(\alpha - m_\lambda)\ \theta(a_{J\lambda} m_\lambda - \alpha)\ \chi_\lambda(\alpha, a_{J\lambda}, b_{J\lambda}), \tag{25}$$

where the parameter $a_{J\lambda}$ determines position of the most distant border of discontinuity, caused by the λ-meson exchange mechanism, and function $\chi_\lambda(\alpha, a_{j\lambda}, b_{J\lambda})$ sets the form of the cut-off. In the given work two models of cut-off are considered: in model A with sharp cut-off $\chi_\lambda(\alpha, a_{J\lambda}, b_{J\lambda}) = 1$, and in the model B with smooth cut-off

$$\chi_\lambda(\alpha, a_{J\lambda}, b_{J\lambda}) = \theta(b_{J\lambda} m_\lambda - \alpha) + \frac{a_{J\lambda}^2 m_\lambda^2 - \alpha^2}{m_\lambda^2 (a_{J\lambda}^2 - b_{J\lambda}^2)}\ \theta(\alpha - b_{J\lambda} m_\lambda). \tag{26}$$

To take into account the inelastic channels, we have modified the Lippmann-Shwinger equation having entered in it the parameter of inelasticity $R_l(q)$

$$T_l(k', k; s) = V_l(k, k') + \frac{2}{\pi} \int_0^\infty \frac{V_l(k', q) T_l(q, k; s)}{q^2 - s - i\eta} R_l(q) q^2 dq. \tag{27}$$

The value of parameter $R_l(q)$ was coordinated with the data of the phase shift analysis in the region of energy of incident nucleon in laboratory frame up to 3 GeV.

Table. Meson coupling constants, masses and cut-off parameters.

λ	π	ρ	σ	ω	η	a_0
m_λ, MeV	138.03	769	550	782.6	548.8	983
$\frac{g^2_{\lambda NN}}{4\pi}$	14.0	0.84 [$\kappa_\rho = 6.0$]	8 (12 in $^3S_1$-state)	5	1.73	2.5
$a_{J\lambda}$	6	2	2	2	2	1.8
$b_{J\lambda}$	4	1.5	1.5	1.5	1.5	1.5

Using the parameters given in Table (see also Ref. [2]), we have calculated the potentials (represented in Figs.1,2) and phase-shifts (Figs.1,2) for NN-scattering in $^1S_0$ and $^3S_1$ - states according the model A with sharp cut-off (solid curves) and the model B with smooth cut-off (dotted curves). For comparison in Fig.1 we have represented the Reid soft-core potential [7]. Data of energy-dependent partial-wave analysis (circles) and single-energy analysis (squares) are taken from Ref. [8]. We note that, for NN-scattering in the $^3S_1$-state, the coupling of the S- and D-wave amplitudes is taken into account in present work through the inelasticity parameter $R_0(q)-$ that is, scattering in the $^3D_1$-wave state is considered as an inelastic channel with respect to scattering in the $^3S_1$-wave state.

5 Conclusion

The new relativistic approach to constructing the effective local operators of NN-interaction, based on the requirement of the proper analytical structure of partial-wave scattering amplitudes is suggested. We determine effective potentials as such local operators in the partial-wave Lippmann-Shwinger equation, which generates relativistic (Feynman) on-mass-shell scattering amplitudes having required singularities on dynamic cuts. The discontinuities of partial-wave scattering amplitudes are determined by model-independent characteristics - renormalized coupling constants and amplitudes of elementary subprocesses with on-mass-shell particles. It can be calculated by methods of the relativistic quantum field theory with the help of various dynamic approaches. In particular, for calculation of the discontinuities of the partial-wave scattering amplitudes on the sites of dynamic cuts nearest to physical region the chiral effective theory can be used.

The method is applied to constructing effective NN-potentials in $^1S_0$- and $^3S_1$-states. It is shown that in our approach the nonlinear in meson-nucleon coupling constants contributions generate the additional repulsion at small distances, not related to ω-meson exchange mechanism. Using the potentials thus obtained in momentum representation, we have calculated the energy dependence of partial-wave phase shifts. The theoretical predictions are in good agreement with available experimental data at kinetic energies of incident nucleon up to $1.5 \div 2$ GeV.

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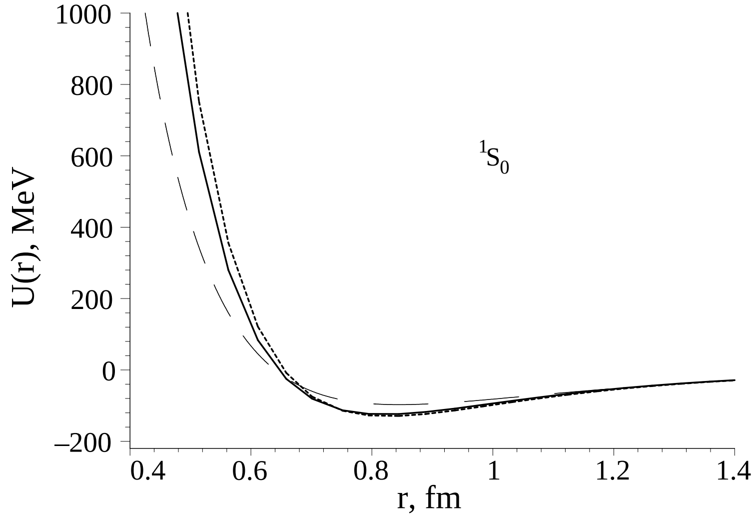

Figure 1: NN-potential $U(r) = V(r)/m_N$ in $^1S_0$-state according the model A (solid curve) and the model B (dotted curve). Dashed curve represents the Raid soft-core potential from Ref. [7].

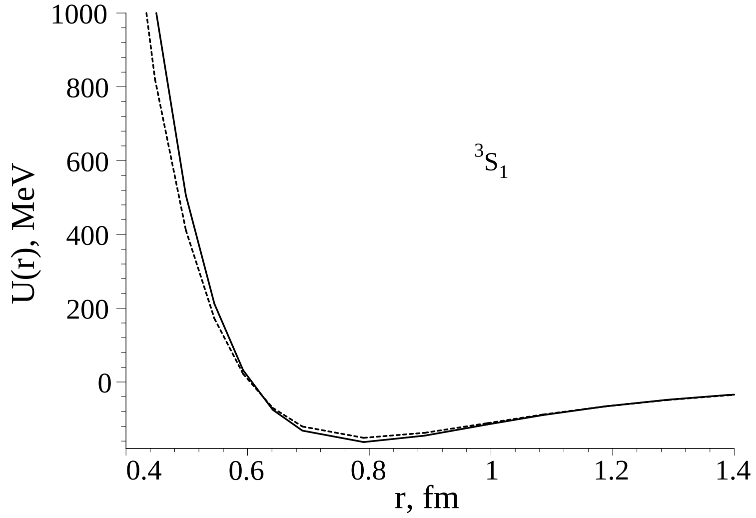

Figure 2: NN-potential $U(r) = V(r)/m_N$ in $^3S_1$-state according the model A (solid curve) and the model B (dotted curve).

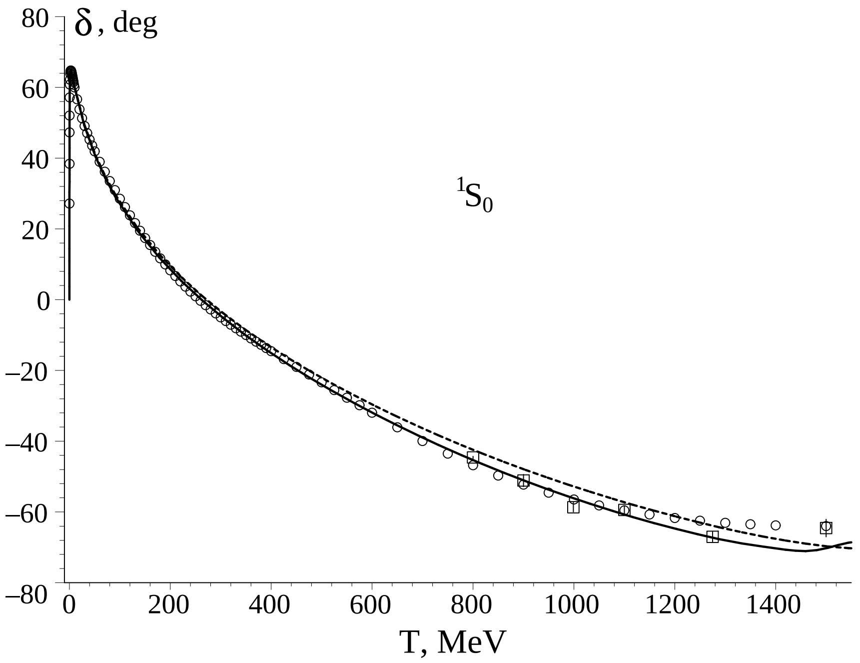

Figure 3: Phase-shifts for NN-scattering in $^1S_0$-state according the model A with sharp cut-off (solid curve) and the model B with smooth cut-off (dotted curve). Data of energy-dependent partial-wave analysis (circles) and single-energy analysis (squares) from Ref. [8].

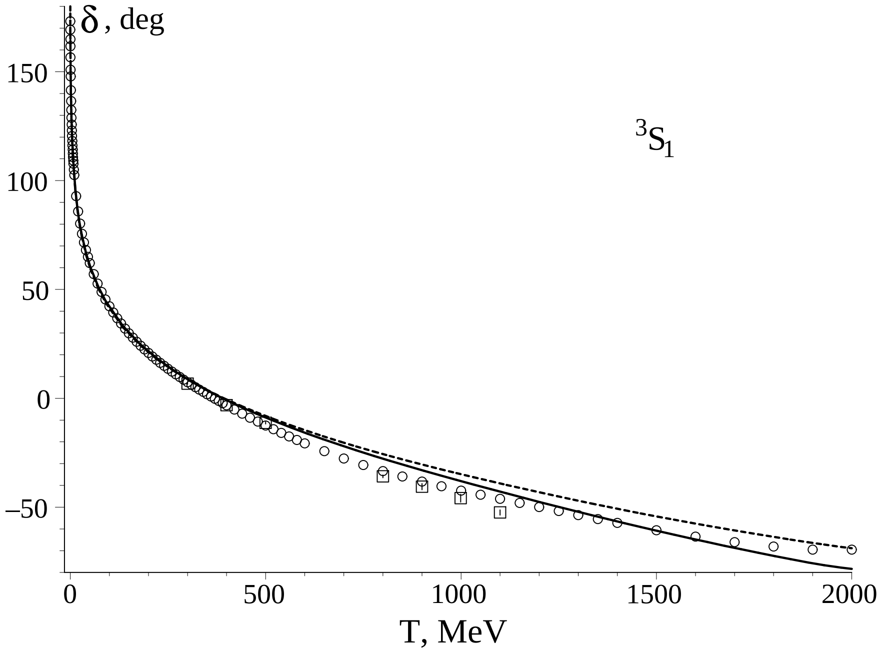

Figure 4: Phase-shifts for NN-scattering in $^3S_1$-state according the model A with sharp cut-off (solid curve) and the model B with smooth cut-off (dotted curve). Data of energy-dependent partial-wave analysis (circles) and single-energy analysis (squares) from Ref. [8].