Construction of singlevalued and multivalued solutions for nonintegrable systems

S. Yu. Vernov
Skobeltsyn Institute of Nuclear Physics
Moscow State University,
Vorob'evy Gory, 119992 Moscow, Russia.
svernov@theory.sinp.msu.ru

Abstract

The Painlevé test is very useful to construct not only the Laurent-series solutions but also the elliptic and trigonometric ones. Such single-valued functions are solutions of some polynomial first order differential equations. The standard methods for the search of the elliptic solutions consist of two independent steps: transformation of a nonlinear polynomial differential equation into a nonlinear algebraic system and finding of solutions of the obtained system. It has been demonstrated by the example of the generalized Hénon–Heiles system that the use of the Laurent-series solutions of the initial differential equation assists to solve the obtained algebraic system and, thereby, simplifies the search of elliptic solutions. This procedure has been automatized and generalized on some type of multivalued solutions. To find solutions of the initial differential equation in the form of the Laurent or Puiseux series we use the Painlevé test. This test can also assist to solve the inverse problem: to find the form of a polynomial potential, which corresponds to the required type of solutions. We consider the five–dimensional gravitational model with a scalar field to demonstrate this.

1 Introduction

The investigations of exact special solutions of nonintegrable systems play an important role in the study of nonlinear physical phenomena. There are a few methods to construct solutions in terms of rational, hyperbolic, trigonometric or elliptic functions (all such functions are solutions of the first order polynomial differential equations) [1–8]. These methods use information about the dominant behavior of solutions in the neighbourhood of their singular points, but do not use the Laurent series representations of the initial system solutions. In [9] a new method has been developed as an alternative way to construct elliptic and elementary solutions and it has been shown that the use of the Laurent series solutions allows to extend the set of solutions [10]. The Laurent series solutions give the information about the global behavior of the differential system and assist to look for not only global solutions but also the first integrals [11].

The direct algebraic method is the substitution of the first order equation, which solutions are elementary or elliptic functions, into the initial differential system to transform it in a nonlinear algebraic system in coefficients of this first order equation and parameters of the initial system. It has been proved that any finite system of algebraic equations can be solved by the Gröbner (Groebner) basis method [12], but calculations for a sufficiently complex algebraic system can be very difficult and expensive. In this paper we show that the use of the Laurent series solutions gives additional algebraic equations and allows to simplify calculations. These equations are linear in coefficients of the first order equation and nonlinear, maybe even nonpolynomial, in parameters of the initial system. Note that one maybe needs to fix some of these parameters to construct the Laurent series solutions. Therefore, in contrast to the Gröbner basis method, the additional equations may be not consequences of the initial algebraic system.

To manifest the main properties of this method we will consider the generalized Hénon–Heiles system, for which analytic and Laurent series solutions have been found in [6, 8, 13].

2 The Painlevé analysis

We interpret solutions of a system of ordinary differential equations (ODE's) as analytic functions, maybe with isolated singular points. A singular point of a solution is said *critical* (as opposed to *noncritical*) if the solution is multivalued (single-valued) in its neighbourhood and *movable* if its location depends on initial conditions [14]. A singular point is said algebraic if the solution has a finite number of values in its neighbourhood. In the opposite case a singular point is either transcendental or essentially singular one.

The general solution of an ODE of order N is the set of all solutions mentioned in the existence theorem of Cauchy, i.e. determined by the initial values. It depends on N arbitrary independent constants. *A special solution* is any solution obtained from the general solution by giving values to the arbitrary constants. A system of ODE's has ***the Painlevé property*** if its general solution has no movable critical singular point [15].

The Painlevé test is any algorithm, which checks some necessary conditions for a differential equation to have the Painlevé property. The original algorithm, developed by P. Painlevé and used by him to find all the second order ODE's with the Painlevé property, is known as the α-method. The method of S.V. Kovalevskaya is not as general as the α–method, but much more simple. In 1980, developing the Kovalevskaya method further, M.J. Ablowitz, A. Ramani and H. Segur [16] constructed a new algorithm of the Painlevé test for ODE's. This algorithm appears very useful to find solutions as a formal Laurent series [13]. First of all, it allows to determine the dominant behavior of a solution in the neighbourhood of the singular point t_0. If the solution tends to infinity as $(t-t_0)^\beta$, where β is a negative integer number, then substituting the Laurent series expansions one can transform nonlinear differential equations into a system of linear algebraic equations on coefficients of the Laurent series. All solutions of an autonomous system depend on the parameter t_0, which characterizes the singular point location. If a single-valued solution depends on other parameters, then some coefficients of its Laurent series have to be arbitrary and the corresponding systems should have zero determinants. The numbers of such systems (named *resonances* or *Kovalevskaya exponents*) can be determined due to the Painlevé test. In such a way we obtain solutions only as a formal series, but really we will use only a finite number of this series coefficients, so, we do not need the convergence of this series.

The classical results allow to construct the suitable form of the first order autonomous polynomial equation. The Painlevé theorem [15, 14] proves that solutions of the equation

$$\sum_{k=0}^{m} P_k(y(t),t)y_t^k = 0, \tag{1}$$

where $y_t(t) \equiv \frac{dy(t)}{dt}$, m is a positive integer number and P_k are polynomials in $y(t)$ and analytic functions in t, have neither movable transcendental nor movable essential singular point. The Fuchs theorem [17, 14] shows that if eq. (1) has no critical movable singular points, then the power of $P_k(y)$ is no more than $2m-2k$, in particular, $P_m(y)$ is a constant.

Therefore, the necessary form of a polynomial autonomous first order ODE with the single-valued general solution is

$$\sum_{k=0}^{m}\sum_{j=0}^{2m-2k} h_{jk}\, y^j y_t^k = 0, \qquad h_{0m} = 1, \tag{2}$$

where h_{jk} are constants. The Briot and Bouquet theorem [18, 9] proves that the general solution of (2) is either an elliptic function, or a rational function of $e^{\gamma x}$, γ being some constant, or a rational function of x. Note that the third case is a degeneracy of the second one, which in its turn is a degeneracy of the first one.

3 Methods for constructing of elliptic solutions

3.1 The generalized Hénon–Heiles system

To compare the methods of the elliptic solutions construction let us consider the generalized Hénon–Heiles system with an additional non-polynomial term, which is described by the Hamiltonian:

$$H = \frac{1}{2}\Big(x_t^2 + y_t^2 + \lambda_1 x^2 + \lambda_2 y^2\Big) + x^2 y - \frac{C}{3}\,y^3 + \frac{\mu}{2x^2}$$

and the corresponding system of the motion equations:

$$\left\{\begin{array}{lcl} x_{tt} & = & -\,\lambda_1 x - 2xy + \dfrac{\mu}{x^3}, \\ y_{tt} & = & -\,\lambda_2 y - x^2 + Cy^2, \end{array}\right. \tag{3}$$

where $x_{tt} \equiv \frac{d^2x}{dt^2}$ and $y_{tt} \equiv \frac{d^2y}{dt^2}$, λ_1, λ_2, μ and C are arbitrary numerical parameters. Note that if $\lambda_2 \neq 0$, then one can put $\lambda_2 = sign(\lambda_2)$ without the loss of generality. If $C = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$ and $\mu = 0$, then (3) is the initial Hénon–Heiles system [19].

The function y, solution of system (3), satisfies the following fourth-order polynomial equation:

$$y_{tttt} = (2C-8)y_{tt}y - (4\lambda_1 + \lambda_2)y_{tt} + 2(C+1)y_t^2 + \frac{20C}{3}y^3 + (4C\lambda_1 - 6\lambda_2)y^2 - 4\lambda_1\lambda_2 y - 4H. \quad (4)$$

We note that the energy of the system H is not an arbitrary parameter, but a function of initial data: y_0, y_{0t}, y_{0tt} and y_{0ttt}. The form of this function depends on μ:

$$H = \frac{1}{2}(y_{0t}^2 + y_0^2) - \frac{C}{3}y_0^3 + \left(\frac{\lambda_1}{2} + y_0\right)(Cy_0^2 - \lambda_2 y_0 - y_{0tt}) + \frac{(\lambda_2 y_{0t} + 2Cy_0y_{0t} - y_{0ttt})^2 + \mu}{2(Cy_0^2 - \lambda_2 y_0 - y_{0tt})}. \quad (5)$$

Formula (5) is correct only if $x_0 = Cy_0^2 - \lambda_2 y_0 - y_{0tt} \neq 0$. If $x_0 = 0$, what is possible only at $\mu = 0$, then we can not express x_{0t} through y_0, y_{0t}, y_{0tt} and y_{0ttt}, so H is not a function of the initial data. If $y_{0ttt} = 2Cy_0y_{0t} - \lambda_2 y_{0t}$, then eq. (4) with an arbitrary H corresponds to system (3) with $\mu = 0$, in opposite case eq. (4) does not correspond to system (3).

3.2 The construction of the nonlinear algebraic system

To find a special solution of eq. (4) one can assume that y satisfies some more simple equation. For example, there exist solutions in terms of the Weierstrass elliptic function, which satisfy the following first-order differential equation:

$$y_t^2 = \tilde{A}y^3 + \tilde{B}y^2 + \tilde{C}y + \tilde{D}, \quad (6)$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are some constants. E.I. Timoshkova [6] generalized eq. (6):

$$y_t^2 = A_4y^3 + A_3y^{5/2} + A_2y^2 + A_1y^{3/2} + A_0y + \tilde{D}, \quad (7)$$

(A_j are constants) and found new one-parameter solutions of the Hénon–Heiles system in two nonintegrable cases ($C = -4/3$ or $C = -16/5$). These solutions (i.e. solutions with $A_3 \neq 0$ or $A_1 \neq 0$) are derived only at $\tilde{D} = 0$, therefore, substitution $y(t) = \varrho(t)^2$ gives:

$$\varrho_t^2 = \frac{1}{4}(A_4\varrho^4 + A_3\varrho^3 + A_2\varrho^2 + A_1\varrho + A_0). \quad (8)$$

Following [8], we use the substitution $y(t) = \varrho(t)^2 + P_0$, where P_0 is a constant, and transform eq. (4) into

$$\begin{aligned} \varrho_{tttt}\varrho &= -4\varrho_{ttt}\varrho_t - 3\varrho_{tt}^2 + 2(C-4)\varrho_{tt}\varrho^3 + (2P_0(C-4) - 4\lambda_1 - \lambda_2)\varrho_{tt}\varrho + 2(3C-2)\varrho_t^2\varrho^2 + \\ &+ (2CP_0 - 4\lambda_1 - 8P_0 - \lambda_2)\varrho_t^2 + \tfrac{10}{3}C\varrho^6 + (2C\lambda_1 + 10CP_0 - 3\lambda_2)\varrho^4 - 2H + \\ &+ 2(2\lambda_1CP_0 + 5CP_0^2 - \lambda_1\lambda_2 - 3P_0\lambda_2)\varrho^2 + \tfrac{10}{3}CP_0^3 + 2\lambda_1CP_0^2 - 3P_0^2\lambda_2 - 2\lambda_1\lambda_2P_0. \end{aligned} \quad (9)$$

If $\varrho(t)$ satisfies (8), then eq. (9) is equivalent to the following system

$$\left\{ \begin{array}{l} (3A_4+4)(2C-3A_4) = 0, \\ A_3(9C - 21A_4 - 16) = 0, \\ 96A_4CP_0 - 240A_4A_2 - 192A_4\lambda_1 - 384A_4P_0 - 48A_4\lambda_2 - \\ \quad - 105A_3^2 + 128A_2C - 192A_2 + 128C\lambda_1 + 640CP_0 - 192\lambda_2 = 0, \\ 40A_3CP_0 - 90A_4A_1 - 65A_3A_2 - 80A_3\lambda_1 - 160A_3P_0 - 20A_3\lambda_2 + 56CA_1 - 64A_1 = 0, \\ 16A_2CP_0 - 36A_4A_0 - 21A_3A_1 - 8A_2^2 - 32A_2\lambda_1 - 64A_2P_0 - 8\lambda_2A_2 + 24CA_0 + \\ \quad + 64\lambda_1CP_0 + 160CP_0^2 - 16A_0 - 32\lambda_1\lambda_2 - 96P_0\lambda_2 = 0, \\ 10A_3A_0 + (5A_2 + 8CP_0 - 16\lambda_1 - 32P_0 - 4\lambda_2)A_1 = 0, \end{array} \right. \quad (10)$$

and the equation for the energy H:

$$\begin{aligned} H &= \frac{1}{384}\Big(-48A_2A_0 + 96CA_0P_0 + 384C\lambda_1P_0^2 + 640CP_0^3 - 9A_1^2 - \\ &- 192A_0\lambda_1 - 384A_0P_0 - 48A_0\lambda_2 - 384\lambda_1\lambda_2P_0 - 576\lambda_2P_0^2\Big). \end{aligned} \quad (11)$$

System (10) has been solved by REDUCE using the standard function **solve** and the Gröbner basis method [8]. The goal of this paper is to show that the use of the Laurent series solutions allows to obtain some solutions of (10) solving only the linear system and nonlinear equation in one variable.

We can not use this method for arbitrary C, because the Laurent series solutions are different for different C. So, first of all we have to fix value of C. Note that solutions with $A_3 = A_5 = 0$ are known, because in this case y satisfies eq. (6) with $\tilde{D} = 0$. The remained problem can be separated into two cases $A_5 \neq 0$ or $A_3 \neq 0$ and $A_5 = 0$. If $A_5 \neq 0$, then from two first equations of system (10) it follows:

$$C = -\frac{4}{3} \quad \text{and} \quad A_4 = -\frac{4}{3} \qquad \text{or} \qquad C = -\frac{16}{5} \quad \text{and} \quad A_4 = -\frac{32}{15}.$$

Surely, if some solution with $A_5 = 0$ corresponds the above-mentioned values of C, then it can be found as well.

3.3 Construction of linear algebraic system

Let choose $C = -4/3$ (the case $C = -16/5$ one can consider analogously) and construct the Laurent series solutions for eq. (9). The method of construction of the Laurent series solution for the generalized Hénon–Heiles system has been described in detail in [13]. For eq. (9) with $C = -4/3$ we obtain that solutions has singularities proportional to $1/t$ and the values of resonances are -1 (corresponds to an arbitrary parameter t_0), 1, 4 and 10. The Laurent series solutions are (we put $t_0 = 0$):

$$\tilde{\rho} = \pm\left(\frac{i\sqrt{3}}{t} + c_0 + \frac{i\sqrt{3}}{24}\left(3\lambda_2 - 2\lambda_1 + 4P_0 + 62c_0^2\right)t + \dots\right),$$

where

$$c_0 = \frac{\pm\sqrt{161700\lambda_1 - 121275\lambda_2 \pm \sqrt{1155(5481\lambda_2^2 - 12768\lambda_1\lambda_2 + 8512\lambda_1^2)}}}{2310}. \qquad (12)$$

Two signs " $\pm$ " in (12) are independent. At the same time, functions $\tilde{\rho}$ and $-\tilde{\rho}$ correspond to one and the same function $\tilde{y}$, so there are four different Laurent series solutions. The coefficients c_3 and c_9 are arbitrary.

The algorithm of the construction of elliptic solutions from the Laurent-series solutions is the following:

- Choose a positive integer m and define the first order ODE (2) with unknown coefficients h_{jk}.
- Compute coefficients of the Laurent series $\tilde{\rho}$. The number of coefficients has to be greater than the number of unknowns.
- Substituting the obtained coefficients, transform eq. (2) into a linear and overdetermined system in h_{jk} with coefficients depending on arbitrary parameters.
- Exclude h_{jk} and obtain the nonlinear system in parameters.
- Solve the obtained system.

To obtain the explicit form of the elliptic function, which satisfies the known first order ODE, one can use the classical method due to Poincaré, which has been implemented in Maple as the package "algcurves" [20].

On the first step we choose eq. (2), which coincides with eq. (8). It means that $m = 2$, all h_{j1} are equal to zero and all $h_{j0} = -A_j/4$. After the second and the third steps we obtain a linear system on A_{j0}. The package of computer algebra procedures, which transforms the first order equation into a such system of algebraic equations has been written in Maple [21].

The obtained system has the triangular form and linear in H, c_3 and c_9 as well. From the first equation we obtain anew that $A_4 = -4/3$. From the second equation it follows

$$A_3 = \frac{16}{3}c_0, \qquad \text{and so on:}$$

$$A_2 = -70c_0^2 - 3\lambda_2 + 2\lambda_1 - 4P_0,$$
$$A_1 = \left(\frac{40}{3}P_0 - 60\lambda_1 + 50\lambda_2 + 1300c_0^2\right)c_0,$$
$$A_0 = -40i\sqrt{3}c_3 - \frac{21535}{12}c_0^4 + \left(\frac{565}{6}\lambda_1 - \frac{405}{4}\lambda_2 - \frac{245}{3}P_0\right)c_0^2 + \frac{7}{4}\lambda_1\lambda_2 - \frac{21}{16}\lambda_2^2 - \frac{7}{12}\lambda_1^2 + \frac{7}{3}\lambda_1 P_0 - \frac{7}{2}\lambda_2 P_0 - \frac{7}{3}P_0^2.$$

From the next equation of the system we obtain c_3 and, finally,

$$A_0 = \frac{15645}{4}c_0^4 + \left(\frac{1545}{4}\lambda_2 - 465P_0 - \frac{1495}{2}\lambda_1\right)c_0^2 + \frac{537}{20}\lambda_1^2 - \frac{663}{20}\lambda_1\lambda_2 + \frac{729}{80}\lambda_2^2 + 19\lambda_1 P_0 - \frac{37}{2}\lambda_2 P_0 - \frac{17}{3}P_0^2.$$

Substituting the values of A_k, which correspond to one of the possible values of c_0, in system (10) we obtain that it is satisfied for all values of λ_1, λ_2 and P_0, so we do not need to solve the nonlinear equations. Therefore we find a solution of the nonlinear algebraic system (10) solving only linear equations and nonlinear equation in one variable. We have used the values of only six coefficients of the Laurent series solutions. Note that, for c_0 and $-c_0$ we obtain one and the same values of A_4, A_2 and A_0, and the opposite values of A_3 and A_1. From (8) it follows that these solutions correspond to $\pm\rho(t)$, and, hence, give one and the same function $y(t)$. Therefore not four, but two different elliptic (or degenerated elliptic) solutions of eq. (4) have been found. The elliptic solutions $y(t)$ are the fourth-order elliptic functions and can be expressed in terms of the Weierstrass elliptic function $\wp(t-t_0)$:

$$y(t-t_0) = \left(\frac{a\wp(t-t_0)+b}{c\wp(t-t_0)+d}\right)^2 + P_0, \tag{13}$$

where constants a, b, c, d and periods of $\wp(t)$ are determined by A_k. The parameters t_0 and P_0, which defines the energy of the system, are arbitrary. Solutions of this type exist in both above-mentioned nonintegrable cases: $C = -16/5$ and $C = -4/3$. Full list of solutions is given in [8].

4 Multivalued solutions

From the Painlevé theorem it follows that a solution of an autonomous polynomial first order differential equation (1) can have only two types of singular points: poles and algebraic branch point. So, the most general type of a formal series solution is the Puiseux-series one. Let us generalize the concerned method on the search of solutions, which can be expanded in the following Puiseux series

$$y = \sum_{k=-L}^{\infty} S_{k/q} t^{k/q}. \tag{14}$$

We seek solutions as a polynomial

$$y = P_L\varrho^L + P_{L-1}\varrho^{L-1} + P_{L-2}\varrho^{L-2} + P_{L-3}\varrho^{L-3} + \ldots + P_0.$$

From (14) it follows that

$$\varrho = \sum_{j=-1}^{\infty} C_{j/q} t^{j/q}.$$

We assume that $\varrho(t)$ satisfies the following equation:

$$\sum_{k=0}^{m}\ \sum_{j=0}^{(q+1)(m-k)} h_{jk}\varrho^j {\varrho_t}^k = 0, \quad h_{0m} = 1, \tag{15}$$

where q is a natural number.

To simplify calculations we can put $P_L = 1$ and $P_{L-1} = 0$ without the loss of generality, because

$$\tilde{\varrho} = \left(\varrho - \frac{P_{n-1}}{n}\right) / \sqrt[n]{P_n}$$

satisfies (15) as well. The construction of the corresponding algebraic system has been automatized in Maple [22]. We plan to automatize similar construction for an arbitrary autonomous polynomial first order equation.

5 Five–dimensional gravitational model with a scalar field

To show how the analysis of singular behavior of solutions can assist to find the form of potential, let us consider the model of gravity field interacting with a single scalar field in five-dimensional space-time [23, 24]. The action is

$$S=\int_M d^4xdr\sqrt{|det\, g_{\mu\nu}|}\left(-\frac{R}{4}+\frac{1}{2}(\partial\phi)^2-V(\phi)\right), \qquad (16)$$

where M is the full five-dimensional space-time. The most general metric with four-dimensional Poincaré symmetry is

$$ds^2=e^{2A(r)}(dx_0^2-dx_1^2-dx_2^2-dx_3^2)-dr^2.$$

If the scalar field depends on the additional coordinate only: $\phi=\phi(r)$, then the independent equations of motion are

$$\left\{\begin{array}{lcl} J' & = & -\dfrac{2}{3}(\phi')^2,\\ J^2 & = & -\dfrac{1}{3}V(\phi)+\dfrac{1}{6}(\phi')^2, \end{array}\right. \qquad (17)$$

where $J\equiv A'\equiv\frac{dA}{dr}$ and $\phi'\equiv\frac{d\phi}{dr}$. The similar systems of the motion equations arise also in the cosmological models with the spatially flat Friedman metric and a scalar or phantom scalar field [25].

Let us analyse the correspondence between $\phi(r)$ and the potential $V(\phi)$. If at singular point

$$\phi(r)\sim\frac{1}{r^m},$$

then from (17) it follows that

$$J'\sim\frac{1}{r^{2m+2}}\qquad\Longrightarrow\qquad J\sim\frac{1}{r^{2m+1}}\qquad\Longrightarrow\qquad V(\phi)\simeq V\left(\frac{1}{r^m}\right)\sim\frac{1}{r^{4m+2}}.$$

It means that solutions with poles proportional to $1/r$ can be obtained only if the power of the polynomial potential $V(\phi)$ is equal to six[1]. For example, let $\phi(r)=\tanh(r)$, for real r this function has no singular point, but system (17) is autonomous, so if $\phi(r)$ is a solution, then $\phi(r-r_0)$, where r_0 is an arbitrary complex constant, is a solution as well. Surely, there exists a such r_0 that $\tanh(r-r_0)$ tends to infinity as $r\to 0$, hence, the function $\tanh(r)$ cannot be a solution for any forth order polynomial potential. In [23] the explicit form of the sixth order polynomial potential $V(\phi)$, which corresponds to $\tanh(r)$ has been found. Analogously one can show that if solutions tend to infinity as $1/r^2$, then the power of $V(\phi)$ is equal to 5. If solutions tend to infinity as $1/r^k$, where k is a natural number greater than two, then $V(\phi)$ can not be a polynomial. In conclusion of this section we say a few words about explicit solutions. Following [23] we assume that $J(r)$ is a function of ϕ:

$$J(r)=-\frac{1}{3}W(\phi).$$

It is straightforward to verify that system (17) is equivalent to

$$\frac{d\phi(r)}{dr}=\frac{1}{2}\,\frac{dW(\phi)}{d\phi}, \qquad (18)$$

$$\left(\frac{dW(\phi)}{d\phi}\right)^2-\frac{1}{3}W(\phi)^2-V(\phi)=0. \qquad (19)$$

Unfortunately the nonautonomous equation (19) can not be solved analytically. For polynomial $V(\phi)$ it is possible, if possible, to find only special solutions: $W(\phi)$ in the polynomial form[2].

Contrary to a scalar field theory without gravitational field, there is not one to one correspondence between the form of the scalar field $\phi(r-r_0)$ and potential $V(\phi)$. The form of the scalar field is defined by $\frac{dW}{d\phi}$, so one can add a constant to $W(\phi)$ and obtain new $V(\phi)$ for the same $\phi(r)$. On the other hand, for given $V(\phi)$ we have not one-, but two-parameter set of functions $\phi(r)$.

[1]Similar solutions can exist, surely, for nonpolynomial potentials as well.

[2]For example, $W(\phi)$ can not be a polynomial if $V(\phi)=(\phi^2-1)^2$ and we do not know a solution for this potential.

6 Conclusion

The Laurent series solutions are useful to find elliptic or elementary solutions in the analytic form. The method, proposed in [9], has been automatized and generalized on some type of multivalued solutions. It converts the local information into the global one and can be used not only as an alternative of the standard method, but also as an addition to it, which assists to find solutions of the obtained algebraic system. We have demonstrated that one can find elliptic solutions of the generalized Hénon–Heiles system solving only linear equations and nonlinear equations in one variable, instead of nonlinear system (10). At the same time, to use this method one has to know not only an algebraic system, but also differential equations, from which this system has been obtained. To find solutions of the initial ODE in the form of the Laurent or Puiseux series we use the Painlevé test. This test can also assist to solve the inverse problem: to find the power of a polynomial potential, which corresponds to the required type of solutions.

The author is grateful to I.Ya. Aref'eva, R. Conte, A.S. Koshelev, M.N. Smolyakov and I.P. Volobuev for valuable discussions. This work has been supported by Russian Federation President's Grant NSh–1685.2003.2 and by the grants of the scientific Program "Universities of Russia" 02.03.028 and 02.02.503.

References

[1] J. Weiss, Phys. Lett. A102 (1984) 329; Phys. Lett. A105 (1984) 387
[2] G.S. Santos, J. of the Physical Society of Japan 58 (1989) 4301
[3] R. Conte, M. Musette, J. Phys. A25 (1992) 5609
[4] V.A. Antonov, E.I. Timoshkova, Russ. Astron. J. 70 (1993) 265
[5] A.V. Porubov, D.F. Parker, Wave Motion 29 (1999) 97
[6] E.I. Timoshkova, Russ. Astron. J. 76 (1999) 470, Astr. Rep. 43 (1999) 406
[7] E. Fan, J. of Phys. A **36** (2003) 7009–7026
[8] E.I. Timoshkova, S.Yu. Vernov, math-ph/0402049, 2004.
[9] R. Conte, M. Musette, Physica D181 (2003) 70, nlin.PS/0302051.
[10] R. Conte, M. Musette, nlin.PS/0407026, 2004.
[11] N.A. Kudryashov, nlin.SI/0408041, 2004.
[12] J.H. Davenport, Y. Siret, E. Tournier, *Calcul Formel, Systemes et Algorithmes de Manipulations Algebriques*, (Masson, Paris, New York, 1987).
[13] S.Yu. Vernov, TMF 135 (2003) 409, Theor. Math. Phys. 135 (2003) 792, math-ph/0209063, 2002.
[14] V.V. Golubev, Lectures on Analytical Theory of Differential Equations, (Gostekhizdat (State Pub. House), Moscow–Leningrad, 1950) {in Russian}.
[15] P. Painlevé, Leçons sur la théorie analytique des équations différentielles, profeesées à Stockholm (septembre, octobre, novembre 1895) sur l'invitation de S. M. le roi de Suède et de Norwège (Hermann, Paris, 1897); Reprinted in: Œuvres de Paul Painlevé, V. 1 (ed. du CNRS, Paris, 1973). On-line version: The Cornell Library Historical Mathematics Monographs, `http://historical.library.cornell.edu/`
[16] M.J. Ablowitz, A. Ramani, H. Segur, J. Math. Phys. 21 (1980) 715 & 1006
[17] von L. Fuchs, *Gesammelte mathematische Werke von L. Fuchs. Hrsg. von Richard Fuchs und Ludwig Schlesinger* (Berlin, Mayer & Müller, 1904–1909). On-line version: The Cornell Library of Historical Mathematics Monographs, `http://historical.library.cornell.edu/`
[18] C.A.A. Briot, J.C. Bouquet, *Theórie des fonctions elliptiques. Deuxiéme édition,* (Gauthier-Villars, Imprimeur-Libraire, Paris, 1875). On-line version: The Cornell Library Historical Mathematics Monographs, `http://historical.library.cornell.edu/`
[19] M. Hénon, C. Heiles, Astron. J. 69 (1964) 73
[20] M. van Hoeij, Package 'algcurves' for Maple V (1997), `http://www.math.fsu.edu/~hoeij/`
[21] S.Yu. Vernov, in: *Proc. of the International Conference "Computer Algebra in Scientific Computing" (CASC'04, St. Petersburg, Russia, 2004)*, eds. V.G. Ganzha, E.W. Mayr, E.V. Vorozhtsov, (Technische Universitat, Munchen, Garching, 2004) p. 457, nlin.SI/0407062, 2004.
[22] S.Yu. Vernov, Preprint SINP MSU 2004–18/757, `http://www.npi.msu.su/science.php3`
[23] O. DeWolfe, D.Z. Freedman, S.S. Gubser, A. Karch, Phys.Rev. D62 (2000) 046008, hep-th/9909134
[24] M. Gremm, Phys.Lett. B478 (2000) 434, hep-th/9912060; Phys. Rev. D62 (2000) 044017, hep-th/0002040.
[25] I.Ya. Aref'eva, A.S. Koshelev, S.Yu. Vernov, astro-ph/0412619, 2004.