Quasielliptic orbits in the dilaton gravity in the newtonian limit

E. G. Vorontsova, G. S. Sharov
Tver State University,
Sadovyj per., 35, 170002, Tver, Russia

Abstract

In the dilaton gravitational model stationary Schwarzschild type solutions (centrally symmetrical solutions in vacuum) are obtained and investigated. Parameters of quasielliptic orbits of test particles are determined in the limit of weak gravitational field.

The considered dilaton gravitational model is the consequence of the string theory in the low-energy limit [1, 2]. The equations of evolution for this model

$$R_{\mu\nu} + 2\phi_{;\mu\nu} = 0, \qquad R + 4(\nabla\phi)^2 = 0. \tag{1}$$

are obtained from the action [1] via variation of the gravitational field $g_{\mu\nu}$ and the dilaton field ϕ accordingly. Here $R_{\mu\nu}$ is the Ricci tensor in $n+1$ - dimensional pseudoriemann manifold $\mathcal{M}_{1,n}$, $R = R^\mu_\mu$ is the scalar curvature.

We search stationary Schwarzschild-type solutions of Eqs. (1), that is centrally symmetrical solutions in vacuum. So the metric in the pseudoriemannian spherically symmetric manifold $\mathcal{M}_{1,n}$ with signature $+,-,-,\dots$ may be taken in the form

$$\begin{gathered} ds^2 = e^{2\alpha(r)}dt^2 - e^{2\beta(r)}dr^2 - r^2 d\Omega^2, \\ d\Omega^2 = d\theta_1^{\,2} + \cos^2\theta_1 d\theta_2^{\,2} + \ldots + \cos^2\theta_1 \ldots \cdot \cos^2\theta_{n-2} d\theta_{n-1}^{\,2}. \end{gathered} \tag{2}$$

Here t is time-like, r is radial and θ_k are angular coordinates.

When we substitute metric (2) and the corresponding nonzero Ricci tensor components

$$\begin{aligned} &R_{00} = e^{2\alpha-2\beta}[\alpha'' + \alpha'^2 - \alpha'\beta' + (n-1)\alpha'/r], \quad R_{11} = -\alpha'' - \alpha'^2 + \alpha'\beta' + (n-1)\beta'/r, \\ &R_{22} = e^{-2\beta}[\beta' r - \alpha' r - n + 2] + n - 2, \qquad R^3_3 = \ldots = R^n_n = R^2_2. \end{aligned}$$

into the system (1), it reduces to the following four nontrivial equations

$$\begin{aligned} &\alpha'' + \alpha'^2 - \alpha'\beta' + (n-1)\alpha'/r = 2\alpha'\phi', \\ &\alpha'' + \alpha'^2 - \alpha'\beta' - (n-1)\beta'/r = 2\phi'' - 2\beta'\phi', \\ &\beta' - \alpha' + (n-2)(e^{2\beta} - 1)/r + 2\phi' = 0, \\ &\alpha'' + \alpha'^2 - \alpha'\beta' + (n-1)[(\alpha' - \beta')r + \tfrac{1}{2}(n-2)(1 - e^{2\beta})]/r^2 = 2\phi'^2. \end{aligned} \tag{3}$$

We substitute expression for ϕ' from the third equation into the first equation of the system (3) and obtain the following relation:

$$-(n-2)e^{2\beta} = r\alpha''/\alpha' + 1.$$

Combining the first, second and fourth equations of the system (3) and also using expression for α' obtained from the third equation we come to the following relation

$$-(n-2)e^{2\beta} = r\phi''/\phi' + 1.$$

Comparing two latter expressions we see that

$$\phi' = Q\alpha'. \tag{4}$$

Here Q is the arbitrary constant.

For $n \geq 3$ the solution of system (3) is found in the form:

$$\begin{gathered} e^{2\alpha} = e^{2\alpha_0}\left|\frac{\gamma - \gamma_+}{\gamma - \gamma_-}\right|^{1/K}, \quad e^{2\beta} = \left(1 - \frac{\gamma}{\gamma_+}\right)\left(1 - \frac{\gamma}{\gamma_-}\right), \\ r = r_0 \frac{|\gamma - \gamma_+|^{A_+}|\gamma - \gamma_-|^{A_-}}{|\gamma|^{1/(n-2)}}, \quad \phi = Q\alpha(r) + \phi_0. \end{gathered} \tag{5}$$

Where we use the parameter $\gamma = r\alpha'$ and the following notations:

$$K = \sqrt{1 - \frac{4Q(1-Q)}{n-1}}, \quad \gamma_\pm = (n-1)\frac{1-2Q \pm K}{4Q(1-Q)}, \quad A_\pm = \frac{1}{2(n-2)}\left(1 \pm \frac{2Q-1}{K}\right).$$

The solution (5) is the generalization of the classical Schwarzschild solution for the Einstein gravity ($n = 3$):

$$ds^2 = \left(1 - \frac{r_g}{r}\right)dt^2 - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 - r^2(d\theta_1^2 + \cos^2\theta_1 d\theta_2{}^2). \tag{6}$$

In the case $\phi = \text{const}$ equations of dilaton gravity (1) transform to the Einstein equations in vacuum $R_{\mu\nu} = 0$. By virtue of equality (4) this case is realized at $Q = 0$. In the limit $Q \to 0$ the solution (5) pass into expression (6) for $n = 3$ or into the following generalization of the Schwarzschild solution for $n > 3$ [3]:

$$ds^2 = \left[1 - (r_g/r)^{n-2}\right]dt^2 - \left[1 - (r_g/r)^{n-2}\right]^{-1} dr^2 - r^2 d\Omega^2. \tag{7}$$

The Schwarzschild solution is one-parameter family with the parameter $r_g = r_0$ (gravitational radius proportional to the central mass). But in the case of the expression (5) we have the two-parameter family of solutions in the class of metrics (2): the parameter r_0 is the analog of r_g and the additional parameter Q demonstrates involving the dilaton field in this solution. It is interesting that only in the case $Q = 0$, $\phi = \text{const}$ the black hole horizon exists [3].

In Ref. [3] we investigated, that solutions (5) have physical sense only in the domain $0 < \gamma < \gamma_+(Q)$ in the $Q\gamma$ plane. In particular, it follows from the behavior of geodesic lines.

For the analysis of observational manifestations of this dilaton model and Schwarzschild-type solution (5) we consider the limit $r \to \infty$. This limit may be treated as Newtonian limit of this model, that is the limit of weak gravitational field. The limits $r \to \infty$ and $\gamma \to +0$ are equivalent because the relation resulting from Eq. (5) (expression for r):

$$\lim_{\gamma \to +0} r = \lim_{\gamma \to +0} r_0 \frac{|\gamma_+|^{A_+}|\gamma_-|^{A_-}}{|\gamma|^{1/(n-2)}} = \lim_{\gamma \to 0} \frac{r_1}{|\gamma|^{1/(n-2)}} = \infty,$$

here $r_1 = r_0|\gamma_+|^{A_+}|\gamma_-|^{A_-}$. The more detailed analysis of the function $r(\gamma)$ at $\gamma \to +0$ results in the expansion

$$\frac{r_1}{r} \simeq \gamma^{\frac{1}{n-2}}\left[1 - \frac{2(1-2Q)}{(n-2)^2}\gamma + 2\frac{(n-1)^2 - n(3n-4)\,Q(1-Q)}{(n-2)^4(n-1)}\gamma^2\right]. \tag{8}$$

Hence, in the head order of small parameter r_1/r the equality $\gamma \simeq (r_1/r)^{n-2}$ takes place. Substituting this series into Eq. (5) and using the relations

$$\gamma_+ \cdot \gamma_- = \frac{(n-1)(n-2)}{4Q(Q-1)}, \quad \gamma_+ + \gamma_- = \frac{(n-1)(1-2Q)}{2Q(1-Q)}$$

in the limit $\gamma \to +0$ we obtain the following expressions for the metric coefficients $e^{2\alpha}$ and $e^{2\beta}$:

$$e^{2\alpha} \simeq C\left[1 - \frac{2}{n-2}\left(\frac{r_1}{r}\right)^{n-2}\right], \quad e^{2\beta} \simeq 1 + \frac{2(1-2Q)}{n-2}\left(\frac{r_1}{r}\right)^{n-2}. \tag{9}$$

The solutions (5) are asymptotically flat in the limit $r \to \infty$ (the limit of weak gravitational field) and tend to the Schwarzschild solution (7) in case $Q = 0$. Comparing Eqs. (9) and (7) we have the relation for the gravitational radius in dilaton solution (5)

$$r_g = r_1\left(\frac{2}{n-2}\right)^{1/(n-2)}. \tag{10}$$

Using the last equality and series (8) we can express γ via the small parameter $\rho = r_g/r$:

$$\gamma \simeq \frac{1}{2}\rho + \frac{1-2Q}{2}\rho^2 + \frac{4 - 17Q(1-Q)}{8}\rho^3, \qquad \rho \equiv \frac{r_g}{r}. \tag{11}$$

We suppose here and below the space has physical dimensionality $n = 3$.

Substituting Eq. (11) into the formulas (5) for the metric coefficients we obtain the following expansion in the Newtonian limit $r \to \infty$ or $\rho \to 0$:

$$
\begin{aligned}
e^{-2\alpha} &= 1 + \rho + (1-Q)\rho^2 + (1-Q)\big(1 - \tfrac{13}{12}Q\big)\rho^3, \\
e^{-2\beta} &= 1 - (1-2Q)\,\rho + \tfrac{1}{2}Q(1-Q)\rho^2 + \tfrac{1}{4}Q(1-Q)(1-2Q)\rho^3.
\end{aligned}
\tag{12}
$$

We use these expansions for study of quasielliptic orbits in this gravitational field. In particular, we analyze the periastron (perihelion) shift, that is important observational manifestation.

We consider trajectories of test particles in the gravitational field (5) similar to orbits of planets in the solar gravitational field. In the Newtonian theory these orbits are ellipses, but in the Einstein theory for the Schwarzschild solution (6) they are quasi-elliptic. For one revolution their periastron shifts on the angle [4]

$$\Delta\theta \simeq \frac{3\pi r_g}{a(1-\epsilon^2)}. \tag{13}$$

Here $r_g = 2GM/c^2$ is the gravitational radius of the gravity source (Sun), a is the big semi-axis of the ellipse and ϵ of its eccentricity. The speed of light $c = 1$ below.

Trajectories of test particles in the gravitational field (5) are time-like geodesic lines ($ds^2 > 0$). We suppose that these particles do not interact directly with the dilaton field ϕ. We consider the motion of test particles in plane $\theta_1 = \ldots = \theta_{n-2} = 0$. Coordinates r and $\theta \equiv \theta_{n-1}$ determine the position of a point (particle) in the moment t. Equations of corresponding geodesic lines for this metric (2) are

$$\frac{d^2t}{ds^2} + 2\alpha'\frac{dt}{ds}\frac{dr}{ds} = 0, \tag{14}$$

$$\frac{d^2r}{ds^2} + \alpha' e^{2\alpha-2\beta}\left(\frac{dt}{ds}\right)^2 + \beta'\left(\frac{dr}{ds}\right)^2 + re^{-2\beta}\left(\frac{d\theta}{ds}\right)^2 = 0, \tag{15}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{d\theta}{ds}\frac{dr}{ds} = 0, \tag{16}$$

Eq. (15) demonstrates that solutions (5) describe gravitational attraction at $\alpha' > 0$. The “physical domain” is given by the condition $\gamma > 0$ (remind that $\gamma = r\alpha'$). The domain with negative γ results in repulsing.

Integrals of Eqs. (14) and (16) are:

$$\frac{dt}{ds} = Ae^{-2\alpha}, \qquad \frac{d\theta}{ds} = \frac{C_\theta}{r^2}. \tag{17}$$

Here $A = E/m$ and $C_\theta = L/m$ are energy and angular momentum per unit mass for a test particle with the mass m. They are conserving first integrals of system.

When we substitute equalities (17) into the expression for metric (2)

$$ds^2 = e^{2\alpha}dt^2 - e^{2\beta}dr^2 - r^2 d\theta^2$$

for planar motions and exclude value s we obtain the differential equation

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{C_\theta^2}e^{-2\beta}\left(A^2e^{-2\alpha} - 1 - \frac{C_\theta^2}{r^2}\right). \tag{18}$$

This equation determines the trajectory of a test particle. Analysis of Eq. (18) demonstrates that the motion of test particle is finite and quasi-elliptic orbits exists only in the case $A < 1$.

We consider the Newtonian limit $r \to \infty$ and use the asymptotical expansions (12) and inequalities

$$r \gg r_g, \qquad C_\theta \gg r_g, \qquad 1 - A \ll 1. \tag{19}$$

Substituting Egs. (12) with the notation $\rho = r_g/r \ll 1$ we rewrite Eq. (18) in the form

$$\frac{d\rho}{d\theta} = \pm\sqrt{f(\rho)}, \qquad f(\rho) = \frac{r_g^2}{C_\theta^2}\Big\{A^2 - 1 + \big[1 - 2Q(1-A^2)\big]\,\rho\Big\} + B_2\rho^2 + B_3\rho^3 + o(\rho^3), \tag{20}$$

$$B_2 = -1 + Q\left[A^2 - \frac{(1{-}Q)(1{-}A^2)}{2}\right]\frac{r_g^2}{C_\theta^2}, \quad B_3 = 1 - 2Q + Q(1-Q)\left[\frac{17A^2}{12} - \frac{(1{-}2Q)(1{-}A^2)}{4}\right]\frac{r_g^2}{C_\theta^2}.$$

A finite motion is quasi-elliptic one only if the equation $f(\rho) = 0$ has three positive roots ρ_1, ρ_2, ρ_3. We numerate the roots in order of increase. Under conditions (19) their values determined by the approached equalities

$$\rho_1 + \rho_2 \simeq \frac{r_g^2}{C_\theta^2}, \qquad \rho_3 \simeq B_3^{-1}\left(1 - Q\frac{r_g^2}{C_\theta^2}\right) - \frac{r_g^2}{C_\theta^2}. \tag{21}$$

For this motion the following inequalities take place: $\rho_1 \le \rho \le \rho_2$, $\rho_1 < \rho_2 \ll \rho_3 \simeq (1-2Q)^{-1}$.

The periastron shift $\Delta\theta$ for 1 revolution we obtain with the help of the equation (20).

$$\Delta\theta = 2\int\limits_{\rho_1}^{\rho_2} \frac{d\rho}{\sqrt{f(\rho)}} - 2\pi = 2\int\limits_{\rho_1}^{\rho_2} \frac{1}{\sqrt{(\rho_2-\rho)(\rho-\rho_1)}}\left\{\Big[B_3(\rho_3-\rho)\Big]^{-1/2} - 1\right\}d\rho.$$

Calculation of integral with using of Eqs. (21) and conditions (19) results to the following expression in the leading order on small values r_g^2/C_θ^2 and $(1-A^2)$:

$$\Delta\theta \simeq \frac{\pi(3-4Q)}{2}\frac{r_g^2}{C_\theta^2} = \frac{\pi(3-4Q)\,r_g}{a(1-\epsilon^2)}. \tag{22}$$

Here we took into account the relations of ellipse parameters a and ϵ with ρ_1 ρ_2:

$$a = \frac{1}{2}\left(\frac{r_g}{\rho_1} + \frac{r_g}{\rho_2}\right) = \frac{r_g}{2}\frac{\rho_1+\rho_2}{\rho_1\rho_2}, \qquad 1-\epsilon^2 = 1 - \left(\frac{\rho_1^{-1} - \rho_2^{-1}}{\rho_1^{-1} + \rho_2^{-1}}\right)^2 = \frac{4\rho_1\rho_2}{(\rho_1+\rho_2)^2}.$$

So we may conclude that in dilaton gravitational model the important observational parameter of theory — periastron shift $\Delta\theta$ (22) differs by the term $1 - \frac{4}{3}Q$ from the expression for this shift (13) in Einstein's general theory of relativity.

References

[1] E.S. Fradkin, A.A. Tseytlin Phys. Lett. B158 (1985) 316

[2] E.G. Vorontsova, G.S. Sharov Theor. Math. Phys. 123 (2000) 549 (hep-th/9904127)

[3] E.G. Vorontsova, G.S. Sharov, in: *Proceedings of the XV International Workshop on Quantum Field Theory and High Energy Physics (QFTHEP 2000)*, ed.by M.Dubinin, V.Savrin, Moscow, 2001, p.394-398 (hep-th/0101004)

[4] L.D. Landau, E.M. Lifshitz, The classical theory of fields, Nauka, Moscow, 1984.