Approach to solving equations in the ring of formal power series

V.F.Edneral
Moscow State University, Moscow, Russia
E-mail: *edneral@theory.npi.msu.su*,
N.N.Vassiliev
Institute for Theoretical Astronomy,
St.Petersburg, Russia
E-mail:*vasiliev@eimi.imi.ras.ru*

Abstract

The paper briefly describes an approach to solving multivariate systems of equations written in formal power series by using a standard basis technique. Authors apply the Buchberger's algorithm with a usual lexicographic ordering to rewrite the system of equations in the "triangle" form analog. This paper has an experimental status.

1 Introduction

Let us look at a system of equations of the type:

$$\Phi_i(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \sum_{q_1,\ldots,q_n\geq 0} f_{i,q_1,\ldots,q_n}\, x_1^{q_1}\cdots x_n^{q_n} = 0 \qquad i=1,\ldots,m \tag{1}$$

here $\Phi_i(0,\ldots,0)=0$ are formal power series, and we are interested in determining a dimension of a solution manifold and in searching *all* branches of solutions of the system (1) near a zero point, i.e. we search all such series $x_1,\ldots,x_n$ in parameters $\alpha_1,\ldots,\alpha_q$, $1<q<n$ which transform (1) to an identity.

Such problems appear in particular at investigating ODEs by the normal form method [1, 2], when a condition of (pseudo) periodicity of solutions can be represented in the form (1). Another source of such problems is Newton's method of approximating solutions of algebraic and differential equations by series. Problems of a close type often arise at uniformization of algebraic curves also.

In multidimensional case problem (1) is not trivial if an origin point is not a simple critical point, i.e. when it is impossible to use an elimination of variables by the implicit function theorem. For the critical point, where conditions of the implicit function theorem are not satisfied and there are several branches of solutions, we have to dissolve all branches of the solution. The advanced technique of solving (1) is really good only for a low dimensional case. We mean Newton's polygon (polytope) method [3, 5, 1], the method of power transformations [1] and a "sigma-process" [4, 5]. A generalization of the methods above for a high dimensional case is very cumbersome.

The large progress in solving multidimensional algebraic systems was achieved owing to Buchberger's algorithms [6] with the Gröbner bases technique because under lexicographic ordering we can rewrite any multidimensional algebraic system to an equivalent "triangle" form which has at least one equation in minimal numbers of variables in accordance to a dimension of solutions.

The ring of formal power series is the Noetherian ring, so the ideal in this ring should have a finite basis. It would be very useful to have a technique for rewriting system (1) to the system with an element which would allow to use the low dimensional standard technique of branches solving [3, 5, 1, 4]. We try to fit the Gröbner basis technique for the discussed problem. The title "standard bases in the ring of formal power series" has been occupied by H. Hironaka [7] and Th. Becker [8] for purpose of creating the Gröbner basis with a special kind of ordering, when instead of oldest term they use least terms. We use usual ordering of terms and the preliminary name "standard bases in polytopes".

2 "Basic" representation for formal power series

Let us rewrite (1) in the form [9]:

$$\begin{aligned}&\Phi_i(x_1,\ldots,x_n) \stackrel{\text{def}}{=}\\ &\stackrel{\text{def}}{=} \sum_{k=1}^{p} x_1^{q_{k1}}\cdots x_n^{q_{kn}} \cdot \sum_{q_1,\ldots,q_n=0}^{\infty} f_{i,k,q_1,\ldots,q_n}\, x_1^{q_1}\cdots x_n^{q_n} = 0,\\ &i=1,\ldots,m\end{aligned} \tag{2}$$

This representation is not unique and differs from (1) by the finite summation in monomials $x_1^{q_{k1}}\cdots x_n^{q_{kn}}$ with "coefficients" which are infinite series starting with constant terms $f_{i,k,0,\ldots,0} \neq 0$. Note, that these coefficients are invertible elements of the power series ring.

3 Truncated basic representation

On the other hand the first approximation of (2) (and (1)) is the system:

$$\sum_{k=1}^{p} x_1^{q_{k1}}\cdots x_n^{q_{kn}} \cdot f_{i,k,0,\ldots,0} = 0 \qquad i=1,\ldots,m \tag{3}$$

This approximation is unique and very close to a representation of each equation in system (1) by the corresponding edges of Newton's polytope. A difference is that there are some additional elements in (3) in a comparison with edges of real Newton's polytopes, because our polytopes are not convex in general (see for example the case $x^3+x^2y^2+y^3$). By the way we will use the title "basis in polytopes" for the standard basis which can be created with respect to the ideal of polynomials of (2) over set of invertible elements of power series ring as coefficients. We will call approximation (3) by a truncated basic representation.

Buchberger's algorithm can be directly used for the truncated basic approximation treatment, so all coefficients in (3) will be usual constants. Of course the resulting ideal is not equivalent to the ideal generated by the original elements of (1), (2) because we need to keep the approximated form (i.e. to eliminate negligible terms) during each step of the algorithm which contains summation. But this nonequivalent result can be consider as a basis of an ideal with respect of a factor ring of the truncated basic representation for an original ideal.

4 Example

An implementation of the algorithm above has been produced in the LISP language as a part of the NORT package [10] under the REDUCE system [11]. The corresponding procedure recasts an original system in the form (2) and then creates for it the "basis in polytopes" in lexicographic or common degree then lexicographic ordering. We restrict ourselves with the case of truncated power series only.

We have applied the program above to the system from paper [12]:

$$\begin{aligned}p_1 &\stackrel{\text{def}}{=} 25x_1^4 - 25x_2^4 - x_3^4 + 16x_1^2x_3^2 + 40x_1x_2x_3 = 0\\ p_2 &\stackrel{\text{def}}{=} x_1^3x_2 + x_1^2x_3(1+x_2) + x_2x_3 = 0\end{aligned} \tag{4}$$

We will consider p_1 and p_2 as truncated power series. System (4) is already in special representation (2). The all "coefficients" $f\ldots$ here are usual constants except of the parentheses $(1+x_2)$. So the truncated basic representation (the first approximation) for (4) is:

$$\begin{aligned}\tilde{p}_1 &\stackrel{\text{def}}{=} 25x_1^4 - 25x_2^4 - x_3^4 + 16x_1^2x_3^2 + 40x_1x_2x_3 = 0\\ \tilde{p}_2 &\stackrel{\text{def}}{=} x_1^3x_2 + x_1^2x_3 + x_2x_3 = 0\end{aligned} \tag{5}$$

The program creates for the $\{\tilde{p}_1,\tilde{p}_2\}$ ideal the "standard basis in polytopes" in lexicographic ordering $x_1 << x_2 << x_3$ with six elements (it takes 0.05 sec!):

1) x_1^{14}
2) $x_2^4x_1^5 - x_1^9$
3) $x_2^5 + x_2^4x_1^2 - x_2x_1^4 - x_1^6$

4) $x_3x_1^9 + x_2^4x_1^8$
5) $x_3x_2 + x_3x_1^2 + x_2x_1^3$
6) $x_3^4 - 16x_3^2x_1^2 - 40x^3x^2x^1 + 25x_2^4 - 25x_1^4$

The following system of *five* equations in the "triangle" form corresponds to the ideal of (5) with the factor ring by the first element x_1^{14}:
2) $x_1^5(x_1^4 - x_2^4) = o(x_1^{13})$
3) $(x_1^4 - x_2^4)(x_1^2 + x_2) = o(x_1^{13})$
4) $x_1^8(x_1x_3 + x_2^4) = o(x_1^{13})$
5) $x_1^3x_2 + x_3(x_1^2 + x_2) = o(x_1^{13})$
6) $25x_1^4 + 16x_1^2x_3^2 + 40x_1x_2x_3 - 25x_2^4 - x_3^4 = o(x_1^{13})$

Of course the system above does not correspond to the ideal of (4) but it presents the first approximation for a solution of the problem (4) and allows us to distinguish its different branches.

If we parametrize $x_1 = y + o(y)$ the system has two classes of non trivial solutions:

a) $x_2 \sim x_1$, i.e. $x_2 = \pm y + o(y)$ or $x_2 = \pm \imath y + o(y)$, $x_3 = -y^3 + o(y^3)$. Solution for x_3 is defined here by element 4. Elements 5 and 6 may be used for calculations some high terms in $O(y^2)$ but they do not influent on the first approximation at this class;

b) $x_2 \sim x_1^2$, i.e. $x_2 = -y^2 + o(y^2)$ and we should establish from element 2 that negligible value is x_1^6 instead of x_1^{14} as in case a). So the element $25x_2^4$ should be eliminate from element 6 and we will have there the equation in variable $z = x_3/y$: $25 + 16z^2 - 40z - z^4 = 0$. This equation has simple solutions fortunately: $z_{1,2} = -2 \pm \imath$, $z_3 = -5$ and $z_4 = 1$. So we have: $x_3 = (-2 \pm \imath)y + o(y)$, or $x_3 = -5y + o(y)$, or $x_3 = y + o(y)$.

Compare this result with the result from paper [12]. There are two real solution:

$$\begin{aligned} x_1 &= -y - 2y^3 + o(y^3) \\ x_2 &= -y^2 - 2y^4 + o(y^4) \\ x_3 &= -y - \tfrac{26}{3}y^3 + o(y^3) \end{aligned} \tag{6}$$

$$\begin{aligned} x_1 &= -y - \tfrac{4}{5}y^3 + o(y^3) \\ x_2 &= -y^2 - \tfrac{4}{5}y^4 + o(y^4) \\ x_3 &= 5y + \tfrac{52}{15}y^3 + o(y^3) \end{aligned} \tag{7}$$

and 3 couples of complex solutions:

$$\begin{aligned} x_1 &= -y - \tfrac{1}{5}(7 \pm \imath)y^3 + o(y^3) \\ x_2 &= -y^2 - \tfrac{1}{5}(7 \pm \imath)y^4 + o(y^4) \\ x_3 &= (-2 \pm \imath)y + \tfrac{6}{53}(3 \mp 16\imath)y^3 + o(y^3) \end{aligned} \tag{8}$$

$$\begin{aligned} x_2 &= \pm x_1 - \tfrac{7}{5}x_1^2 + o(x_1^2) \\ x_3 &= -x_1^3 \pm x_1^4 + o(x_1^4) \end{aligned} \tag{9}$$

$$\begin{aligned} x_2 &= \pm \imath x_1 + \tfrac{7}{5}x_1^2 + o(x_1^2) \\ x_3 &= -x_1^3 \mp \imath x_1^4 + o(x_1^4) \end{aligned} \tag{10}$$

For a creation of real standard bases for truncated power series of the kind (1) (probably with respect of some a factor ring) we should take into account full length coefficients in the basic representation above in spite of the truncated basic representation.

5 Conclusion

We have tried to apply this approach to a treatment of expressions of type (1),(2) with approximations more high level than the first, i.e. for non constant coefficients $f\ldots$ there. But we can not find at this moment the constructive conditions of convergence of Buchberger's algorithm over this set which is not a field. But experimentally we can conclude that sometimes this convergence takes place, so probably such conditions exist.

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