Quantum Relaxation of $n-$ Level Systems

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Abstract

An interaction between $n-$level atom and a large dissipative system (heat bath) is considered. $SU(n)-$group coherent states as useful and convenient basis are employed. Fokker-Planck - like equations for $\mathcal{P}-$ symbol of reduced density matrix taking into account possible squeezed fluctuations of a heat bath are derived. New formulas for correlators of the transition operators between levels of the system are obtained and influence of squeezing on the shape of radiation lines are discussed.

1 Introduction, Fokker - Planck equations approach

The treatment of a small (dynamical) system interacting with a large dissipative system - (a heat bath) has a long history [1, 2]. Methods of deriving damping equations have been developed by many authors [3, 4, 5, 6, 7]. Usual methods are related with master equations formulation for the dynamical system density matrix. In papers [8, 9] were intimated the importance of squeezing of the bath in quantum optical measurement devices.

Let us introduce a quantum system (multilevel atom, for example) interacting with a large dissipative system (photon heat bath). The model Hamiltonian can be expressed as

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_{int}, \tag{1}$$

where $\hat{H}_A$ is the atomic (dynamical system) Hamiltonian linear in $N-1$ diagonal generators of the $SU(n)$ group (coefficients of the linear combination are determined by energy level distances). $\hat{H}_B$ is the free heat reservoir Hamiltonian. $\hat{H}_{int}$ is the interaction Hamiltonian, which may include the interaction of atom with given external classical field (regular or stochastic).

As usual we use the irreversibility approximation for the full system density operator $\hat{\mathcal{R}}\,(t)$:

$$\hat{\mathcal{R}}\,(t) = \hat{\rho}_A\,(t)\,\otimes\,\hat{\rho}_B\,(0). \tag{2}$$

The master equation for the reduced density matrix $\hat{\rho}(t) = tr_B[\hat{\mathcal{R}}(t)]$ of a $n-$ level atoms system in the interaction picture can be written as ($\hbar = 1$)

$$\begin{aligned}
\frac{\partial \hat{\rho}}{\partial t} = \sum_{a=1}^{M} \frac{1}{2}\gamma_a\,[\,(\mathcal{N}_a + 1)\,(2\,\hat{E}_a^-\,\hat{\rho}\,\hat{E}_a^+ - \hat{E}_a^+\,\hat{E}_a^-\,\hat{\rho} - \hat{\rho}\,\hat{E}_a^+\,\hat{E}_a^-\,) \\
+\mathcal{N}_a\,(\,2\,\hat{E}_a^+\,\hat{\rho}\,\hat{E}_a^- - \hat{E}_a^-\,\hat{E}_a^+\,\hat{\rho} - \hat{\rho}\,\hat{E}_a^-\,\hat{E}_a^+\,) \\
-\mathcal{S}_a\,(\,2\,\hat{E}_a^+\,\hat{\rho}\,\hat{E}_a^+ - \hat{E}_a^+\,\hat{E}_a^+\,\hat{\rho} - \hat{\rho}\,\hat{E}_a^+\,\hat{E}_a^+\,) \\
-\bar{\mathcal{S}}_a\,(\,2\,\hat{E}_a^-\,\hat{\rho}\,\hat{E}_a^- - \hat{E}_a^-\,\hat{E}_a^-\,\hat{\rho} - \hat{\rho}\,\hat{E}_a^-\,\hat{E}_a^-\,)\,].
\end{aligned} \tag{3}$$

In this equation $\hat{E}_a^{\pm}$ are (dipole) transition operators between atomic energy levels and, simultaneously, ladder generators of $n-$ dimensional fundamental representation of the $SU(n)-$ group, a type of transition "a" is in correspondence with numeration of $SU(n)$ algebra basic elements, $M \leq n\,(n-1)/2$ is the number of permissible transitions, γ_a- is a cavity damping constant in the transition of kind "a", $\mathcal{N}_a$ is a mean number of photons of the heat bath and $\mathcal{S}_a$ is a complex squeezing parameter of the photon mode on the frequency of the transition of kind "a".

In the derivation of Eq.(3) we are taking into account the existence of squeezed fluctuations of the bath [10]. For the thermal bath "organized" over a squeezed vacuum state

$$|0>_{sq} = \exp\,[(\bar{\zeta}\hat{b}^2 - \zeta\,\hat{b}^{+2})/2]\,|0>,$$

where $\zeta = r\,e^{i\,\theta}$ is the Stoler's squeezing parameter [12], we get

$$\mathcal{N}_a = [(< n > +1/2)\,\cosh 2\,r_j - 1/2]|_{\omega_j=\omega_a},$$

$$\mathcal{S}_a = [(< n > +1/2)\,e^{i\,\theta_j}\,\sinh 2\,r_j]|_{\omega_j=\omega_a},$$

$< n >$ denotes here a mean number of quanta of usual bath (without squeezing) with the same temperature.

It should be pointed out that same equations hold for the case of cooperative relaxation of N multilevel atoms, but there, operators $\hat{E}_a^{\pm}$ are generators of (reducible) direct product of N copies of $SU(n)$ fundamental representations.

For the following it is important to use the diagonal $SU(n)$ coherent states representation of the reduced density matrix

$$\hat{\rho}(\,t\,) = \int_{\mathcal{X}} d\mu(z\,,\bar{z})\,\mathcal{P}\,(z\,,\bar{z}\,,t)|z><z|, \tag{4}$$

where $\mathcal{X} = SU(n)/G_0$ is a quotient space of the group $SU(n)$ with invariant measure $d\mu$, $\quad G_0 \subset SU(n)$. The subgroup G_0 is determined by the choice of a standard vector $|\psi_0>$ in the Perelomov's definition of the Lie group coherent states [11], (the simplest choice of $|\psi_0>$ as the "vacuum" state with lower energy of a single $n-$ level atom leads to $G_0 = U(n-1)$).

It is interesting to note that Eq.(4) exist in general case of arbitrary density matrix without restrictions in view of finite-dimensional nature of all unitary irreducible representations of the $SU(n)$ group.

Employing the explicit action of operators $E_a^{\pm}$ on coherent states projectors $|z><z|$, we can put the operator equation (3) in the form of classical Fokker-Planck-like equation for the function $\mathcal{P}(z\,,\bar{z}\,,t)$:

$$\frac{\partial}{\partial t}\mathcal{P}\,(z\,,\bar{z}\,,t) = \hat{\mathbf{L}}\,\mathcal{P}\,(z\,,\bar{z}\,,t), \tag{5}$$

where operator $\hat{\mathbf{L}}$ is a partial differential operator of the second order, depending on representation of the $SU(n)$ group. (Particular cases of Eq.(5) valid for the $(2\,J+1)-$ level atom ($SU(2)$ group) and atomic system with symmetry of full symmetrical representation $D(P,0)$ of the $SU(3)$ group would be considered next.)

It is evident that the formal solution of the last equation takes the view [7, 10]

$$\mathcal{P}\,(z,\,\bar{z};t) = \int_{\mathcal{X}} d\mu(z^{'}\,,\bar{z}^{'})\,\mathcal{K}(z,\bar{z};t|z^{'}\,,\bar{z}^{'};t_0)\,\mathcal{P}_0(z^{'}\,,\,\bar{z}^{'};\,t_0), \tag{6}$$

where $\mathcal{P}_0$ is the $P-$ symbol of atom's initial density matrix $\rho_A\,(t_0)$, $\quad \mathcal{K}$ is a Green Function (propagator) of the Fokker-Planck equation, and $z = (z_1,\ldots,z_r)$, r is the (complex) dimension of the coset $SU(n)/G_0$.

In the Fokker-Planck equation approach the problem reduces to calculation of the propagator in an explicit form. As well known, the propagator is a solution of Eq.(5) with singular initial boundary condition

$$\lim_{t\to 0}\,\mathcal{K}(z,\,\bar{z};\,t\,|\,z^{'},\,\bar{z}^{'};\,0) = \delta(z,\,\bar{z};\,z^{'},\,\bar{z}^{'}), \tag{7}$$

here δ is a $\delta-$function defined on the manifold $SU(n)/G_0$. Two possible ways should be employed to accomplish these ends - the method of spherical function expansion (connected with eigenvalue problem for the Laplace-Beltrami operator on coset $SU(n)/G_0$) and the technique of $1/N$ "quasiclassical" asymptotics when Eq.(5) is written for the case of large values of parameter $N \to \infty$ (cooperative relaxation) [10]. In this work we have considered the first approach only.

Suppose we get the explicit form for function $\mathcal{K}$. How to use it for the calculation of physical relevant values?

After some algebra we get for the mean value of some operator of atomic system $\hat{A}(t)$ the following formula: algebra

$$< \hat{A}(t) >= \int < z|\,\hat{A}^0(t)\; > \mathcal{P}(z,\,\bar{z};\,t)\,d\mu(z,\,\bar{z}), \tag{8}$$

Besides, a two-point correlator of operators $\hat{A}$, and $\hat{B}$ is presented as double integral over the coset $\mathcal{X}$

$$< \hat{A}(t)\,\hat{B}(0) >= \int\int\; < z|\hat{A}^0\,|z>\;\mathcal{K}(z,\bar{z};t|z^{'}\,,\bar{z}^{'};0)\,\mathcal{P}_{\hat{B}\,\cdot\,\hat{\rho}(0)}(z^{'}\,,\bar{z}^{'};0)\,d\mu(z,\bar{z})\,d\mu(z^{'}\,,\bar{z}^{'}), \tag{9}$$

here $\hat{A}^0(t) = \hat{U}_0^{-1}(t)\,\hat{A}\,\hat{U}_0(t)$, $\hat{U}_0(t)$ is the evolution operator of free dynamical system, $\mathcal{P}_{\hat{B}\,\cdot\,\hat{\rho}(t)}$ denotes the $P-$ symbol of the product of operators $\hat{B}\,\cdot\,\hat{\rho}(0)$ at the initial moment $t=0$.

The proof of Eqs.(8,9) is somewhat lengthy and will not be presented here. Similar, but more unwieldy, expressions exist for mean values of multi-time ordered operator products.

The advantage of the expression (9) lies in possibility of two-point correlator direct calculation without recourse to the quantum-regression theorem and related methods [6].

Taking the Fourier transform of the correlator $<\hat{E}_a^+(t)\,\hat{E}_a^-(0)>$ calculated according Eq.(9) we are able to receive the line shape function of transition "a" in the atom.

$$g_a(\omega) \sim Re \int_0^\infty e^{-i\omega t} \;<\hat{E}_a^+(t)\hat{E}_a^-(0)>\,dt. \qquad (10)$$

2 Two-level atom in the squeezed bath

The Eq.(5) takes the view:

$$\frac{\partial f}{\partial t} = \frac{\gamma}{2}\{\frac{\partial}{\partial z}\,[(\mathcal{N}+1)(2\,J\,z + \frac{\partial}{\partial z}\,z^2 + \frac{\partial}{\partial \bar{z}}\,z^2\,\bar{z}^2) +$$
$$+\mathcal{N}(-2\,J\,z + \frac{\partial}{\partial z} + z^2\,\frac{\partial}{\partial z}) + \mathcal{S}(\frac{\partial}{\partial z} + 2\,\frac{\partial}{\partial\,\bar{z}}\,\bar{z}^2) + \bar{\mathcal{S}}\,z^2\frac{\partial}{\partial z}\,z^2\;] + c.c.\}\,f, \qquad (11)$$

here $f(z,\bar{z};t) \equiv \mathcal{P}(z,\bar{z};t)/(1+z\,\bar{z})^2$, $z \in SU(2)/U(1)$ is the point of closed complex plane (the stereographic projection of the Bloch sphere).

For getting of Eq.(11) solutions the method of spherical functions expansion is usable in cases of small values of J.

$$f(z,\bar{z}) = \sum_{l=0}^{2J}\sum_{m=-l}^{l} F_{lm}(t)\,Y_{lm}(z,\bar{z})\,(1+z\bar{z})^{-2}. \qquad (12)$$

In variables $z,\bar{z}$ spherical function obeys to equations

$$\left[(1+z\,\bar{z})^2\,\frac{\partial^2}{\partial z\,\partial\bar{z}} + l\,(l+1)\right]\,Y_{lm}(z,\bar{z}) = 0, \qquad (13)$$

(14)

$$(\bar{z}\,\frac{\partial}{\partial\bar{z}} - z\,\frac{\partial}{\partial z})Y_{lm}(z,\bar{z}) = m\,Y_{lm}(z,\bar{z}). \qquad (15)$$

In particular case of two-level atom ($J=1/2$) we find the propagator

$$\mathcal{K}(z,\bar{z};t|z',\bar{z}';0) =$$
$$\frac{1}{\pi} + \frac{3}{\pi}[(1+z\,\bar{z})(1+z'\,\bar{z}')]^{-1}\{2\,(z\,\bar{z}'+\bar{z}\,z')\,[e^{-(\eta+\delta)\,t}\,\cosh\gamma|\mathcal{S}|\,t + e^{-(\eta+\delta)\,t+i\,\Psi}\,\sinh\gamma|\mathcal{S}|\,t\,]$$
$$+(1-z\,\bar{z})(1-z'\,\bar{z}')\,e^{-2\,(\eta+\delta)\,t} + (1-z\bar{z})\,\frac{\eta-\delta}{\eta+\delta}\,(1-e^{-2\,(\eta+\delta)\,t}\,)\,\}, \qquad (16)$$

where $\eta = \frac{\gamma}{2}\,(\mathcal{N}+1)$, $\delta = \frac{\gamma}{2}\,\mathcal{N}$, and Ψ is the phase of bath squeezing parameter $\mathcal{S}$. In the limit $t\to\infty$,

$$\mathcal{K} \to \mathcal{P}_{eq}(z,\,\bar{z}) = \frac{1}{\pi}\left[1 + 3\,\frac{(\eta-\delta)}{(\eta+\delta)}\,\frac{(1-z\,\bar{z})}{(1+z\,\bar{z})}\right],$$

where $\mathcal{P}_{eq}$ is the equilibrium density matrix $P-$ symbol.

A radiation transition line shape, received at a predetermined initial density matrix of atom $\hat{\rho}(0)$, is given by the Fourier transform of the $SU(2)$ ladder operators correlator:

$$g(\omega) \sim Re \int_0^\infty e^{-i\omega t}\;<\hat{J}_+(t)\hat{J}_-(0)>\,dt$$

$$= \frac{1}{2}\,\frac{z_0\,\bar z_0}{1+z_0\,\bar z_0}\;Re\left[\frac{1}{i(\omega-\omega_0)+(\eta+\delta-\gamma|\mathcal{S}|)}+\frac{1}{i(\omega-\omega_0)+(\eta+\delta+\gamma|\mathcal{S}|)}\right]. \qquad (17)$$

In (17) we have chosen $\hat\rho(0) = |z_0 >< z_0|$ (atom prepared in the initial $SU(2)$ coherent state $|z_0 >$). Initial thermal equilibrium density matrix with different from the heat bath temperature also was investigated. But in that case, we have not found in the line shape expression terms depending on squeezing parameter $\mathcal{S}$.

3 Three-level atom driven by a squeezed bath

The three-level atom is a simplest model in quantum optics with nonequidistant spectrum. Consider an atom with energy levels $E_1 < E_2 < E_3$, and $E_2 - E_1 = \omega_{12} \neq \omega_{23} = E_3 - E_1$, $(\hbar = 1)$. As an example, the case of allowed transitions $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$ will be treated.

Passing some details we get the following Fokker-Planck-like equation for the function $f(z_1,\bar z_1; z_2,\bar z_2;t) = \mathcal{P}(z_1,\bar z_1; z_2,\bar z_2;t)\,(1+z_1\,\bar z_1+z_2\,\bar z_2)^{-3}$, $[(z_1,\, z_2) \in SU(3)/U(2)]:$

$$\frac{\partial f}{\partial t} = \hat{\mathbf{L}}\, f. \qquad (18)$$

$$\begin{aligned}
\hat{\mathbf{L}} \equiv {} & \frac{\partial}{\partial z_1}\,[\,\frac{N}{2}\,\frac{(\gamma_1+\gamma_2)\,z_1\,z_2\,\bar z_2}{1+z_1\,\bar z_1+z_2\,\bar z_2}+\eta_1\;z_1\,z_2\,(\frac{\partial}{\partial \bar z_2}\,\bar z_2^2 \\
& +\bar z_2\,\frac{\partial}{\partial \bar z_1}\,\bar z_1)+(\delta_1-\eta_2)\,z_1\,z_2\,\frac{\partial}{\partial z_2}+\delta_2\,z_2\,\bar z_2\,\frac{\partial}{\partial \bar z_1}\,] \\
& +\frac{\partial}{\partial z_2}\,[\frac{N}{2}\,\frac{\gamma_1\,(1+z_2\,\bar z_2)+\gamma_2\,z_1\,\bar z_1\,z_2}{1+z_1\,\bar z_1+z_2\,\bar z_2}+\eta_1\,(z_2^2\,\frac{\partial}{\partial \bar z_2}\,\bar z_2^2+z_2\,\bar z_2\,\frac{\partial}{\partial \bar z_1}\,\bar z_1 \\
& +\frac{\partial}{\partial z_2}\,\bar z_2+z_2\,\frac{\partial}{\partial z_1}\,z_1)+\delta_1\,(\frac{\partial}{\partial z_2}+\frac{\partial}{\partial \bar z_2})-\delta_2\,z_1\,z_2\,\frac{\partial}{\partial z_1}+\eta_2\,z_1\,\bar z_1\,\frac{\partial}{\partial z_2}\,]+c.c.,
\end{aligned} \qquad (19)$$

(Here indices 1 and 2 correspond to transitions $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$, respectively).

The Eq.(3)is written for the general case of full symmetric irreducible representation $D(N,\,0)$ of the group $SU(3)$. For $N = 1$ -(the relaxation of a single three-level atoms) we are calling spherical function expansion [7] on the coset $SU(n)/U(2)$, those functions $\{Y\}$ are the eigenfunctions of the Laplace-Beltrami operator on factor space $SU(3)/U(2)$

$$\bigtriangledown^2\, Y = n(n+2)\, Y,$$

where

$$\bigtriangledown^2 = (1+z_1\,\bar z_1+z_2\,\bar z_2)[(1+z_1\,\bar z_1)\,\frac{\partial^2}{\partial z_1\,\partial \bar z_1}+z_1\,\bar z_2\,\frac{\partial^2}{\partial z_1\,\partial \bar z_2}+\bar z_1\,z_2\,\frac{\partial^2}{\partial \bar z_1\,\partial z_2}+(1+\bar z_2\,z_2)\,\frac{\partial^2}{\partial \bar z_2\,\partial z_2}].$$

After tedious calculations the line radiation shape on the transition $1 \leftrightarrow 2$ is written as

$$g_{12}(\omega) \sim Re\int_0^\infty e^{-i\omega t}\;<\hat J_+(t)\hat J_-(0)>\;dt = \sum_{j=1}^{5}\frac{a_j}{(\omega-\omega_{12})^2+b_j^2}. \qquad (20)$$

One can see it is a sum of Lorentz-like terms with parameters:

$$a_1 = 7\,b_1/15,\quad a_2 = 14\,b_2/15,\quad a_3 = 10\,b_3/15,\quad a_4 = c_4\,b_4,\quad a_5 = c_5\,b_5;$$

$$b_1 = \delta_1+\delta_2+\eta_2,\quad b_2 = \delta_1+\delta_2+\eta_1,\quad b_3 = \eta_1+\eta_2+\delta_2,\quad b_4 = |\lambda_1|,\quad b_5 = |\lambda_2|;$$

$$c_4 = \frac{21\lambda_1+14\eta_1+22(\eta_2+\delta_2)}{30(\lambda_1-\lambda_2)}-\frac{5\gamma_1\lambda_1-\gamma_2\lambda_1+(\gamma_1-2\gamma_2)\eta_1+(\gamma_1+\gamma_2)(2\eta_2+2\delta_2-\eta_1+4\gamma_2)}{9\lambda_1(\lambda_1-\lambda_2)},$$

$$c_5 = \frac{21\lambda_2+14\eta_1+22(\eta_2+\delta_2)}{30(\lambda_2-\lambda_1)}-\frac{5\gamma_1\lambda_2-\gamma_2\lambda_1+(\gamma_1-2\gamma_2)\eta_1+(\gamma_1+\gamma_2)(2\eta_2+2\delta_2-\eta_1+4\gamma_2)}{9\lambda_2(\lambda_2-\lambda_1)},$$

$$\lambda_{1,2} = -[3\eta_1+2(\eta_2+\delta_2)]/2 \pm \{([3\eta_1+2\eta_2+\delta_2)]^2/4-2[\eta_1(\eta_1+2\eta_2+2\delta_2)+\eta_1(\gamma_2-\eta_1)]\}^{1/2}.$$

Similar expressions exist for line shape on transition $2 \to 3$.

4 Conclusions

It was shown that the Fokker - Planck equations method is particularly advantageous for solutions of quantum kinetic problem. Detailed analysis of multi-time correlators and line shape expressions in cases of two-, three- and multilevel atoms (including the interaction with external (stochastic) fields and cooperative relaxation) would be discussed elsewhere. Special interest will be paid to cases of the relaxation of specially prepared superpositions of initial quantum states both for atomic subsystem and heat bath.

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