Dual Symmetry in Gravitation

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Abstract

An Abelian two-parameter dual group is found to exist in Gravitation. Irreducible representations of the group are obtained: the curvature tensor was expanded in the sum of twice anti-self-dual and self-dual parts. Gravitational instantons are defined as (real) solutions of the usual duality equations. Central-symmetry solutions of these equations are obtained. The twice anti-self-dual part of the curvature tensor admits the use to introduce a new gravitational equations generalizing the Einstein equations. However, the theory obtained reduces to the conformal-flat Nordströem theory.

1 Dual Group in Gravitation

We refer to [1] for more details about the continuous dual symmetry in gauge theories.

First we need to introduce some notations and recall some facts writing them down in a convenient form for the further use.

The curvature tensor $R_{\mu\nu\rho\sigma}$ in gravitation is an analog of the electromagnetic strength. There are two channels (two pairs of indices) not to be quite independent for which the dual conjugation operation can be introduced. So

$$*R_{ijkl} = \frac{1}{2}E_{ijmn}R^{mn}{}_{kl}, \quad R*_{ijkl} = \frac{1}{2}R_{ij}{}^{mn}E_{mnkl}, \quad E_{ijkl} = \frac{1}{\sqrt{-g}}\varepsilon_{ijkl}$$

The properties

$$**\mathbf{R} = \mathbf{R}** = -\mathbf{R}$$

can be easily verified.

The essential properties of the curvature tensor can be rewritten in terms of the right-handed and/or the left-handed dual-conjugate tensor.

Often one considers the so-called twice dual curvature tensor (TDCT)

$$*R*_{\mu\nu\rho\sigma} = \frac{1}{4}E_{\mu\nu\alpha\beta}R^{\alpha\beta\gamma\delta}E_{\gamma\delta\rho\sigma}.$$

Using the expression for the antisymmetric ε-symbols product in terms of the Kroneker δ-symbols [2] it is easy to express TDCT in terms of the curvature tensor

$$*R*^{\mu\nu}{}_{\rho\sigma} = -R^{\mu\nu}{}_{\rho\sigma} + \delta^{\mu}_{\sigma}R^{\nu}_{\rho} + \delta^{\mu}_{\rho}R^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}R^{\nu}_{\rho} - \delta^{\nu}_{\rho}R^{\mu}_{\sigma} + \frac{1}{2}R(\delta^{\nu}_{\rho}\delta^{\mu}_{\sigma} - \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}). \tag{1}$$

By means of TDCT the curvature tensor in a natural and invariant manner is expanded in a sum of two irreducible parts

$$R_{\mu\nu\rho\sigma} = \mathcal{R}_{\mu\nu\rho\sigma} + \mathcal{S}_{\mu\nu\rho\sigma}$$

where

$$\mathcal{R}_{\mu\nu\rho\sigma} = \frac{1}{2}(R_{\mu\nu\rho\sigma} - *R*_{\mu\nu\rho\sigma}) \quad \mathcal{S}_{\mu\nu\rho\sigma} = \frac{1}{2}(R_{\mu\nu\rho\sigma} + *R*_{\mu\nu\rho\sigma}).$$

It is easy to verify the following properties are take place (the tensor indices are omited)

$$*\mathcal{R} = \mathcal{R}*, \quad *\mathcal{S} = -\mathcal{S}*, \quad *\mathcal{R}* = **\mathcal{R} = \mathcal{R}** = -\mathcal{R}, \quad *\mathcal{S}* = \mathcal{S} = -**\mathcal{S} = -\mathcal{S}**.$$

For instance

$$*\mathcal{R} = *\frac{1}{2}(\mathbf{R} - *\mathbf{R}*) = \frac{1}{2}(*\mathbf{R} + \mathbf{R}*) = \mathcal{R}*, \quad *\mathcal{S} = *\frac{1}{2}(\mathbf{R} + *\mathbf{R}*) = \frac{1}{2}(*\mathbf{R} - \mathbf{R}*) = -\mathcal{S}*$$

etc. Since tensors $\mathcal{R}$ and $\mathcal{S}$ transforms simply in the dual conjugate manner, the notations can be improved. We take into account that among the left-handed and right-handed dual-conjugate tensors (e.g. $*\mathcal{R}, \mathcal{R}*$) only one independent tensor

$$*\mathcal{R} = \mathcal{R}* = \overset{*}{\mathcal{R}}, \qquad *\mathcal{S} = -\mathcal{S}* = \overset{*}{\mathcal{S}}.$$

Using the explicit expression for TDCT, (1) one can write down an explicit expression for $\mathcal{R}$ and $\mathcal{S}$

$$\mathcal{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}\left(g_{\mu\sigma}R_{\nu\rho} + g_{\nu\rho}R_{\mu\sigma} - g_{\mu\rho}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\rho}\right) + \frac{R}{4}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) =$$

$$= C_{\mu\nu\rho\sigma} + \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{2}$$

$$\mathcal{S}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma}\right) + \frac{R}{4}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})$$

where $C_{\mu\nu\rho\sigma}$ is the conformal Weil tensor. It is also easy to obtain

$$\mathcal{R}_{\mu\nu}{}^{\mu}{}_{\sigma} = \frac{R}{4}g_{\nu\sigma}, \qquad \mathcal{R}_{\mu\nu}{}^{\mu\nu} = R, \qquad \mathcal{S}_{\mu\nu}{}^{\mu}{}_{\sigma} = R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}, \qquad \mathcal{S}_{\mu\nu}{}^{\mu\nu} = 0 \tag{3}$$

Now one gets ready to introduce a dual group in gravitation. Let us consider the transformation

$$\mathbf{R} \to \mathbf{R}': \quad \mathbf{R}' = e^{*\alpha}\mathbf{R}e^{*\beta} \tag{4}$$

where α and β are independent dual angles. More precisely, the symbolic notation (4) means

$$\mathbf{R}' = \mathbf{R}\cos\alpha\cos\beta + *\mathbf{R}\sin\alpha\cos\beta + \mathbf{R}*\cos\alpha\sin\beta + *\mathbf{R}*\sin\alpha\sin\beta \tag{5}$$

Quite clear, the transformations of the type (5) forms an Abelian two-parameter group.

Next one finds that the tensors $\mathcal{R}, \mathcal{S}$ transform *simply* when acted by the group (5)

$$\mathcal{R}' = e^{*(\alpha+\beta)}\mathcal{R} = \mathcal{R}e^{*(\alpha+\beta)}, \quad \mathcal{S}' = e^{*(\alpha-\beta)}\mathcal{S} = \mathcal{S}e^{-*(\alpha-B)} \tag{6}$$

and actually realize an irreducible representation of the dual group in gravitation.

By analogy with the dual electrodynamics [1], as a consequence of (6) we immediately obtain two dual gravitational invariants

$$I_1 = (\mathcal{R}^2)^2 + (\overset{*}{\mathcal{R}}\mathcal{R})^2, \quad I_2 = (\mathcal{S}^2)^2 + (\overset{*}{\mathcal{S}}\mathcal{S})^2 .$$

Here $\mathcal{R}^2 = \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$ etc. Perhaps, it will be more convenient to use the linear combinations of these invariants which are expressed in terms of the curvature tensor

$$J_1 = (\mathbf{R}^2)^2 + (*\mathbf{RR})^2 + (\mathbf{RR}*)^2 + (*\mathbf{RR}*)^2, \quad J_2 = \mathbf{R}^2(*\mathbf{RR}*) - (*\mathbf{RR})(\mathbf{RR}*)$$

where in fact

$$*\mathbf{RR} = \mathbf{RR}*, \quad *\mathbf{RR}* = \mathbf{R}(*\mathbf{R}*)$$

2 Gravitational Instantons

2.1 Discussion

The current state of the gravitational instantons question seems to be somewhat intricate. The Penrose's [3] instantons (nonlinear gravitons) are the (anti-)self-dual complex solutions of the Einstein equations. Hawking introduces the instantons as Euclidean solutions to Einstein's equations with finite action [4]. This subject is reviewed e.g. in [5]. Somehow or other gravitational instantons are related to solutions of the Einstein equations in the *Euclidean* space and should obey the (anti-)self-duality equations to be understood as follows

$$R_{\mu\nu} = \lambda g_{\mu\nu}, \qquad *C_{\mu\nu\rho\sigma} = \pm C_{\mu\nu\rho\sigma}$$

where $C_{\mu\nu\rho\sigma}$ is the Weil tensor. The instantons do not appear, to a certain degree, to depend on dynamics. They are most probably reflects deep kinematic-topological properties. For the Einstein gravitation it is not a point.

It is worthwhile to relate instantons to any dynamical equations, in particular to the Einstein's ones. These equations as distinct from the Yang-Mills equations, place too hard restrictions on the curvature tensor from point of view the (anti-)self-duality properties.

Let us call as the gravitational instantons the solutions to the duality equations in *pseudo-Euclidean* space for the tensors $\mathcal{R}$ and S

$$\overset{*}{\mathcal{R}}= \pm i\mathcal{R}, \qquad \overset{*}{\mathcal{S}}= \pm i\mathcal{S} \tag{7}$$

These equations are quite equivalent to the duality conditions in the Yang-Mills theory. Of course, the **R**-space is real and equations (7) are reduced to

$$\mathcal{R}_{\mu\nu\rho\sigma} = 0, \qquad \mathcal{S}_{\mu\nu\rho\sigma} = 0 \tag{8}$$

So, the real gravitational instantons are determined by equations (8). To avoid misunderstanding it should be emphasized that the equations (8) must not be regarded as a system (set).

Having been written in other form, such equations are given in the book [6] and they were obtained otherwise . Solutions to these equations (which are not obtained and are not presented in the book) are referred to as twice (anti-)self-dual ones, which are similar to usual instantons by their properties.

It follows from our approach that they are usual gravitational instantons.

Next a few central-symmetry solutions are shown.

2.2 The 4-central-symmetric solutions

The Metric Choice Let us look for the solutions to (8) as a metric of the form

$$ds^2 = e^{\nu(\rho)}d\rho^2 - \rho^2[d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2)]$$

where

$$ds_0^2 = d\rho^2 - \rho^2[d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2)]$$

4-spherical flat metric. Calculating the curvature tensor gives four nonzero (diagonal) components of the tensor $R_{\mu\nu}$ and six nonzero (diagonal) components of the tensors $R_{\mu\nu\rho\sigma}$.

$\mathcal{R}_{\mu\nu\rho\sigma} = \mathbf{0}$ Six components of the equation not to be turned out by identities reduce to the only that of the *first* order, with solution

$$\nu'(\rho) = \frac{2}{\rho}(1 - e^{\nu}), \qquad e^{\nu(\rho)} = \frac{\rho^2}{\rho^2 - C}$$

The metric is getting flat if $\rho \to \infty$ or $C = 0$

$\mathcal{S}_{\mu\nu\rho\sigma} = \mathbf{0}$ Six components of the equation not to be reduced to identities reduce to a differential *first* order equation alone with solution

$$\nu'(\rho) = \frac{2}{\rho}(e^{\nu} - 1), \qquad e^{\nu} = \frac{1}{1 - C\rho^2}$$

If $\rho \to 0$ or $C = 0$ the metric gets flat.

$\mathbf{R}_{\mu\nu} = \mathbf{0}$ Einstein's equations in empty space have a trivial solution only: $e^{\nu} = 0$. This merely emphasizes the fact that the gravitational instantons are poorly compatible to Einstein's equations. The solutions described at this section are quite analogous to the spherical-symmetric BPST's instanton [7]

2.3 The Static Central-Symmetric Solutions

The Metric Choice We search for a solution as follows (the metric is just as in Landau and Lifshits [2])

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

As a result of the curvature tensor calculation it turned out four nonzero (diagonal) components of the tensor $R_{\mu\nu}$ and six nonzero (diagonal) ones of $R_{\mu\nu\rho\sigma}$.

$\mathcal{R}_{\mu\nu\rho\sigma} = \mathbf{0}$ Six equations not to be reduced to identities reduce to the only differential equation of the *second* order

$$\lambda = \nu, \qquad \nu''(r) = \frac{2}{r^2}(e^\nu - 1)$$

This is rather nontrivial equation. It can be exact solved [8]. The solution is represented in two forms

$$e^\nu = \frac{C_1^2}{2} r^2 \sin^{-2}[\frac{C_1}{\sqrt{2}}(r - C_2)]$$

or

$$e^\nu = \frac{C_1^2}{2} r^2 \sinh^{-2}[\frac{C_1}{\sqrt{2}}(r - C_2)]$$

With the constant C having been equal to zero, the solution become more simple

$$e^\nu = \frac{r^2}{(r - C)^2}$$

This solution is asymptotically flat in tending $r \to \infty$ and for a small r

$$e^\nu \sim r^2, \qquad r \to 0$$

$\mathcal{S}_{\mu\nu\rho\sigma} = \mathbf{0}$ Six equations not to be identities reduce to the only equation of the *second* order

$$\lambda = -\nu, \qquad \nu''(r) + \nu'^2(r) = \frac{2}{r^2}(1 - e^{-\nu})$$

which is solved simply

$$e^\nu = 1 + C_1 r^2 + \frac{C_2}{r}$$

It is seen Schwartzschild's solution is contained in this that (if $C_1 = 0$). For large r the first metric coefficient goes to infinity, what may point out the metric to be closed.

2.4 Generalized Instantons

By analogy with the dual electrodynamics and non-Abelian theory, the generalized instantons are defined as follows (it is not the equations system)

$$\mathcal{R}_{\mu\nu\rho\sigma} \cos\varphi + \overset{*}{\mathcal{R}}_{\mu\nu\rho\sigma} \sin\varphi = 0 \tag{9}$$

$$\mathcal{S}_{\mu\nu\rho\sigma} \cos\psi + \overset{*}{\mathcal{S}}_{\mu\nu\rho\sigma} \sin\psi = 0 \tag{10}$$

where

$$\cos\varphi = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin\varphi = \frac{b}{\sqrt{a^2 + b^2}} \quad \cos\psi = \frac{c}{\sqrt{c^2 + d^2}}, \quad \sin\psi = \frac{d}{\sqrt{c^2 + d^2}}$$

$$a = \mathcal{R}^2, \quad b = \mathcal{R}\overset{*}{\mathcal{R}}, \quad c = \mathcal{S}^2, \quad d = \mathcal{S}\overset{*}{\mathcal{S}}$$

It follows from (9),(10)

$$a^2 + b^2 = 0, \qquad c^2 + d^2 = 0$$

or $a = 0$, $b = 0$, $c = 0$, $d = 0$. The instantons $\mathcal{R} = 0$ and $\mathcal{S} = 0$ are easily seen to obey the equations (9),(10). Perhaps, there exists some way to complexify **R**-space, so that the instanton notion in gravitation would be more comprehensive as it should be in non-Abelian theory.

3 The Gravitation Equations

If there are tensor to possess the basic symmetries of the curvature tensor, metric and the energy-momentum tensor of matter, new gravitational equations could be constructed. Let us begin from the trivial but visual example. It is possible to construct tensor by means of the metric and the energy-momentum tensor which has the curvature tensor symmetries. The following equation is postulated

$$R_{\mu\nu\rho\sigma} = const(g_{\mu\rho}T_{\nu\sigma} + g_{\nu\sigma}T_{\mu\rho} - g_{\mu\sigma}T_{\nu\rho} - g_{\nu\rho}T_{\mu\sigma}) \tag{11}$$

Is this equation good or bad? It is bad as it follows from below. Let the energy-momentum tensor be concentrated at finite range of the space. Out of the range the equation is given by

$$R_{\mu\nu\rho\sigma} = 0$$

Thus, the equation (11) predicts the absence of gravity wherever the matter is absent.

Let us try one more to find the gravitational equation using twice anti-self-dual part of the curvature tensor

$$\mathcal{R}_{\mu\nu\rho\sigma} = A(g_{\mu\rho}T_{\nu\sigma} + g_{\nu\sigma}T_{\mu\rho} - g_{\mu\sigma}T_{\nu\rho} - g_{\nu\rho}T_{\mu\sigma}) \tag{12}$$

One can show that in the long run the equation reduces to the conformal flat Nordstrem's theory [9]. Really, contracting it with respect to indices μ and ρ also taking into account (3) we obtain

$$\frac{1}{4}Rg_{\nu\sigma} = A(2T_{\nu\sigma} + g_{\nu\sigma}T) \tag{13}$$

One more contraction gives

$$R = 6AT \tag{14}$$

Expressing $T_{\nu\sigma}$ from (13) and substituting it to (12), also taking into account (14),(2) we find in the end

$$\mathcal{R}_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) = \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

Thus, $C_{\mu\nu\rho\sigma} = 0$. Together with (14) this equation is formulation of Nordstrem's conformal flat theory which for example predicts no deviation of light in the gravity field.

Being consequent, we have to try constructing equation by means of the tensor $\mathcal{S}$

$$\mathcal{S}_{\mu\nu\rho\sigma} = B(g_{\mu\rho}T_{\nu\sigma} + g_{\nu\sigma}T_{\mu\rho} - g_{\mu\sigma}T_{\nu\rho} - g_{\nu\rho}T_{\mu\sigma})$$

One can show this equation reduces to that

$$R_{\mu\nu} - \frac{R}{4}g_{\mu\nu} = 2BT_{\mu\nu}, \qquad T = 0$$

It would be of interest to consider the system: electromagnetic field – gravitation starting from these equations rather than the Maxwell-Einstein ones.

4 Duality and Variational Principle in Gravitation

We concerned the dynamical aspects just in the previous subsection trying to construct gravitational equations by means of the dual symmetric parts of the curvature tensor only. It is of interest to establish a variational principle which is compatible to duality. As it was noted the Einstein equations and, consequently, the Hilbert variational principle is poor compatible to duality.

However, a direct attempt to create a gravitational theory with the dual symmetry to be involved faces troubles.

Let us treat the problem by keeping an analogy to electrodynamics. It is known that the quadratic Lagrangian equations of electrodynamics are dual invariant. In the gravitational case that all would have been analogous if the equations had taken place

$$R_{\mu\nu\rho}{}^{\sigma}{}_{;\sigma} = 0 \qquad R*_{\mu\nu\rho}{}^{\sigma}{}_{;\sigma} \equiv 0 \tag{15}$$

But for the quadratic in $\mathbf{R}$ Lagrangian one obtains the equation

$$R_{\mu(\nu\rho)}{}^{\sigma}{}_{;\sigma} = 0 \tag{16}$$

rather than the first one of (15). Christoffel's symbols are suggested to be related to metric in the usual way but connections are varied rather than the metric.

However, the left-hand-side of (16) is identically zero on account of the circular transposition identity. Thus we have no variational principle leading to the equations (15) and at this point the analogy to electrodynamics already vanishes.

Perhaps, one should consider *nonsymmetric* connections introducing in this way torsion. In any case it would enable to avoid the equation (16). None forbid however to compound the dual symmetric objects by means of the tensors $\mathcal{R}$ and $\mathcal{S}$. In effect it has been done in the subsection about the gravitational instantons. It is possible also the direct use of two gravitational dual invariants. We have to repeat however that besides the problem to choose the Lagrangian, there is another difficulty. If the basic dynamical variables are symmetric connections related to the metric as usual then the dual invariant Lagrangian theory will be ugly in view of (16).

5 Conclusion

In the paper an attempt to consider consequences of the theory with the incorporated dual symmetry is made. The dual symmetry in gravitation has been investigated as well. Such an approach to gravitation has not appeared to take place before. The dual electrodynamics is an elegant nonlinear model which might be related to the linear Maxwell electrodynamics. The local dual angle arises in a natural way in the framework of this theory. This angle enables to introduce the local dual transformations. By means of these transformations the theory may be reformulated in terms of the magnetic charge. Solutions of the considered equations were referred to as the generalized instantons (the BPST-instanton obeys these equations). Nontrivial gravitational instantons are found. Irreducible representations of the dual group in gravitation give new opportunities to establish new gravitational equations. The problem related to variational principle notion for the dual symmetric gravitation is discussed.

We would like to note the next tasks of interest. It is necessary to establish a dual symmetric variational principle in gravitation. Perhaps, the torsion should be included into the theory to obtain a self-consistent theory.

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