Study of Wilson loop functionals in 2D Yang-Mills theories

J.M. Aroca
Departament de Matemàtica Aplicada i Telemàtica, Universitat Politècnica de Catalunya,
C./ Jordi Girona 1 i 3, Campus Nord
08034 Barcelona, Spain
and
Yuri Kubyshin
Institute for Nuclear Physics, Moscow State University
119899 Moscow, Russia

Abstract

The derivation of the explicit formula for the vacuum expectation value of the Wilson loop functional for an arbitrary gauge group on an arbitrary orientable two-dimensional manifold is considered both in the continuum case and on the lattice. A contribution to this quantity, coming from the space of invariant connections, is also analyzed and is shown to be similar to the contribution of the monopoles.

1 Introduction

Yang-Mills theory in two dimensions has been an object of intensive studies during almost two decades. It is known that the classical theory is trivial and the quantum theory has no propagating degrees of freedom. However, this theory possesses very many interesting properties when formulated on topologically nontrivial spaces [1] and also in the large N limit (N is related to the rank of the gauge group $G = SU(N)$) [2]. A 2D Yang-Mills theory is almost topological in the sense that, for example, for compact spacetime manifolds of area (two-dimensional volume) V its partition function depends only on e^2V, where e is the gauge coupling constant. Also, as we will see, the vacuum expectation value of a Wilson loop depends only on the area of the region surrounded by the loop and is independent of the points where the loop is located. This is a manifestation of the invariance under the area preserving diffeomorphisms. It is believed that 2D Yang-Mills theories share some of the important qualitative features of 4D ones, in particular the area law behaviour. Recently there has been a revival of interest in studying 2D Yang-Mills theories because it was shown that in the strong coupling limit they can be related to lower-dimensional strings [3] and because of the special role such theories play in M-theory [4]. Rich mathematical structures appearing in two-dimensional gauge theories were studied in a number of papers [1], [5] - [7] (see lecture [8] for a review and references). The partition function and the vacuum expectation values of Wilson loop functionals were calculated by many authors with various techniques [9] - [11]. Physical aspects of these theories, in particular the analysis of Polyakov loops and the θ-vacua were discussed in Refs. [12].

In the present contribution we analyze vacuum expectation values of the Wilson loop functionals. As a (Euclidean) space-time a smooth two-dimensional orientable Riemannian manifold M will be considered. The gauge group G is supposed to be a compact Lie group. The gauge potential A_μ can be characterized by the 1-form $A = A_\mu dx^\mu$ on M associated with the connection form on the principal fibre bundle $P(M, G)$ [13] in a standard way via local sections (see Refs. [14] for geometrical desciption of gauge theories). We denote by $\mathcal{A}$ the space of smooth connections.

Let us fix a point x_0 in M and consider based loops defined in a standard way, as continuous mappings of the unit interval $I = [0, 1]$ into M:

$$\gamma : I \to M, \quad s \to \gamma(s) \in M, \quad s \in I$$

with $\gamma(0) = \gamma(1) = x_0$. For a given $A \in \mathcal{A}$ we associate with each loop γ the element of G called holonomy

$$H_\gamma(A) = \mathcal{P} \exp\left(ie \oint_\gamma A \right), \tag{1}$$

where $\mathcal{P}$ means the path ordering. We will be interested in traced holonomies

$$T_\gamma(A) \equiv \frac{1}{d_R} TrR\left(H_\gamma(A)\right) = \frac{1}{d_R} \chi_R(H_\gamma(A)), \tag{2}$$

called also Wilson loop variables [15], where R is an irreducible representation of G, d_R is its dimension and χ_R is its character. Let us denote by $\mathcal{G}$ the group of local gauge transformations, i.e. the group of smooth vertical automorphisms of P. It is known that T_γ suffice to separate points in $\mathcal{A}/\mathcal{G}$ representing classes of gauge equivalent configurations and, therefore, enable to reconstruct all smooth gauge connections up to gauge equivalence [16]. Due to this property the Wilson loop functionals (2) form a natural set of gauge invariant functions of connections in the classical theory.

We will study the vacuum expectation value $< T_\gamma >$ of the Wilson loop functional given by

$$< T_\gamma >= \frac{1}{Z(0)} Z(\gamma), \quad Z(\gamma) = \int \mathcal{D}A e^{-\mathcal{S}} T_\gamma(A), \tag{3}$$

where $\mathcal{S}$ is the Yang-Mills action. In Sect. 2 we will discuss, as an example, the calculation of (3) in the abelian case for arbitrary M. In Sect. 3 $< T_\gamma >$ will be calculated on the lattice for a general gauge group. In Sect. 4 we will study a special class of connections, namely the invariant connections, and their contribution to $< T_\gamma >$ in the case $M = S^2$ and $G = SU(2)$. In Sect. 5 this contribution will be compared with the complete vacuum expectation value (3).

2 Continuum case, $G = U(1)$

The action $\mathcal{S}$ in (3) is given by the standard expression

$$\mathcal{S} = \frac{1}{2} \int_M F \wedge *F = \frac{1}{4} \int_M dv F_{\mu\nu} F^{\mu\nu}.$$

In general the space of connections $\mathcal{A}$ consists of a number of components, or sectors $\mathcal{A}_\alpha$, labelled by elements α of an index set $\mathcal{B}$, $\alpha \in \mathcal{B}$. The functional integral is represented as a sum over the elements of $\mathcal{B}$, each term of the sum being the functional integral over the connections in $\mathcal{A}_\alpha$. This feature has an analog in quantum mechanics: in the case of multiply connected space M evaluation of the functional integral includes summation over the connected components of the space of paths in M and integration over the paths within each component [17].

The set $\mathcal{B}$ of connected components of $\mathcal{A}$ is in 1-1 correspondence with the space of non-equivalent principal G-bundles $P(M, G)$ over the manifold M. Let us denote this space as $\mathcal{B}_G(M)$, $\mathcal{B} \cong \mathcal{B}_G(M)$. The problem of classification of fibre bundles was considered in a number of books and articles. Following closely the lectures by Avis and Isham [18] we obtained that in the case when M is a two-dimensional manifold (actually, even a CW-complex) $\mathcal{B}_G(M)$ is equal to

$$\mathcal{B}_G(M) \cong H^2(M, \pi_1(G)),$$

the second cohomology group of M with coefficients given by elements of the first homotopy group of the gauge group G (see details in [19]). An equivalent classification of principal fibre bundles over a two-dimensional surface in terms of elements of the group Γ, specifying the global structure of G through the relation $G = \tilde{G}/\Gamma$, where $\tilde{G}$ is the universal covering group, was given in the second article of Ref. [1]. For completeness, we present here a list of the first homotopy groups $\pi_1(G)$ for some Lie groups which are of interest in gauge theories: $\pi_1(SU(n)) = \pi_1(Sp(n)) = 0$, $\pi_1(SO(n)) = Z_2$ ($n = 3$ and $n \geq 5$), $\pi_1(U(n)) = Z$.

Let us consider the abelian case. $U(1)$-bundles are classified by elements of the second cohomology group $H^2(M, Z)$. The class $c_1(P) \in H^2(M, Z)$, corresponding to the bundle P, is known as the first (integer) Chern class [13], [20]. For a closed orientable 2-dimensional manifold M $H^2(M, Z) \cong Z$. For the bundle P_n, characterized by the label $n \in \mathcal{B} \cong Z$, the integer cohomology class can be represented by $eF^{(n)}/(2\pi)$ with $F^{(n)}$ being the curvature 2-form defined locally through $F^{(n)} = dA^{(n)}$, where $A^{(n)}$ is the gauge 1-form given by a connection on P_n. The integral

$$\frac{e}{2\pi} \int_M F^{(n)} = n,$$

does not depend on the choice of the connection and gives the 1st Chern number, also called the topological charge.

From the discussion above we conclude that

$$Z(\gamma) = \sum_{n=-\infty}^{\infty} \int \mathcal{D}A^{(n)} e^{-\mathcal{S}(A^{(n)})} T_\gamma(A^{(n)}) = \sum_{n=-\infty}^{\infty} Z^{(n)}(\gamma), \tag{4}$$

where in each term $Z^{(n)}$ the integration goes over the gauge potentials with the Chern number equal to n. We represent $A^{(n)}$ as $A^{(n)} = \tilde{A}^{(n)} + a$, where $\tilde{A}^{(n)}$ is a potential with

$$\int_M \tilde{F}^{(n)} = \frac{2\pi n}{e}, \tag{5}$$

$\tilde{F}^{(n)} = d\tilde{A}^{(n)}$, and a is a 1-form with zero Chern number. As $\tilde{A}^{(n)}$ we take instanton configurations, i.e. solutions of the equation of motion. In the literature they are often referred to as monopoles, and we will follow this terminology in the present paper. Then $Z^{(n)}(\gamma)$ in (4) becomes

$$Z^{(n)}(\gamma) = e^{-\mathcal{S}(\tilde{A}^{(n)})} T_\gamma(\tilde{A}^{(n)}) Z_0(\gamma), \tag{6}$$

$$Z_0(\gamma) = \int_{\mathcal{A}^{(0)}} \mathcal{D}a e^{-\mathcal{S}(a)} T_\gamma(a), \tag{7}$$

and the functional integral in $Z_0(\gamma)$ goes over the connections on the trivial bundle P_0 with zero topological charge.

It turns out that for the evaluation of (6) we do not need explicit expressions for the monopole solutions $\tilde{A}^{(n)}$. Indeed, it is easy to show using Eq. (5) that

$$\mathcal{S}(\tilde{A}^{(n)}) = \frac{2\pi^2 n^2}{Ve^2}.$$

If $\gamma = \partial\sigma$, where σ is the interior of γ and $\tilde{A}^{(n)}$ is regular in σ, then by using Stoke's theorem we obtain that

$$\oint_\gamma \tilde{A}^{(n)} = \oint_{\partial\sigma} \tilde{A}^{(n)} = \int_\sigma d\tilde{A}^{(n)} = \int_\sigma \tilde{F}^{(n)} = \frac{2\pi n}{Ve} S,$$

where we denoted the area of the surface σ by S. Then the contribution of the monopoles to the full function (4) is equal to

$$\begin{aligned} Z_{mon}(\gamma) &:= \sum_{n=-\infty}^{\infty} e^{-\mathcal{S}(\tilde{A}^{(n)})} T_\gamma(\tilde{A}^{(n)}) \\ &= \sum_{n=-\infty}^{\infty} \exp\left(-\frac{2\pi^2 n^2}{Ve^2} + i\frac{2\pi n}{V}\nu S\right) = e\sqrt{\frac{V}{2\pi}} \sum_{l=-\infty}^{\infty} \exp\left[-\frac{Ve^2}{2}\left(\frac{S}{V}\nu + l\right)^2\right], \end{aligned} \tag{8}$$

the last equality was obtained using the Poisson summation formula (we have taken the Wilson loop variable in an arbitrary irreducible representation so that $T_\gamma(A) = \exp\{i\nu e \oint_\gamma A\}$, ν being an integer).

Now let us turn to the calculation of the contribution $Z_0(\gamma)$ given by the functional integral Eq. (7). The power of the exponent in $T_\gamma(A)$ can be understood as a functional J_γ acting on A and written as

$$J_\gamma[A] \equiv e \oint_\gamma A = \int_M dv A_\mu(x) J^\mu_\gamma(x).$$

Then the integral (7) becomes gaussian and can be easily calculated. However, the components J^μ_γ are not smooth functions,

$$J^\mu_\gamma(y) = e \oint_\gamma dx^\mu \frac{1}{\sqrt{|\det g_{\mu\nu}|}} \delta(y - x),$$

so that $J_\gamma[A]$ should be treated as a singular form, or weak form, [21] for the accurate calculation of the contribution of fluctuations around the monopoles. The result is

$$Z_0(\gamma) = \mathcal{N} e^{-\frac{e^2}{2}\nu^2 \frac{S(V-S)}{V}}.$$

Combining this with the contribution of the monopoles (8) we obtain the final expression for the expectation value of the Wilson loop variable for a homologically trivial loop γ in the abelian case:

$$< T_\gamma >= \frac{\sum_{l=-\infty}^{\infty} \exp\left[-\frac{e^2}{2}V\left(\frac{S}{V}\nu + l\right)^2 - \frac{e^2}{2}\nu^2\frac{S(V-S)}{V}\right]}{\sum_{l=-\infty}^{\infty} \exp\left[-\frac{e^2}{2}Vl^2\right]}. \tag{9}$$

The expression (9) depends only on the total area V of the compact two-dimensional manifold M and the area S of σ surrounded by γ. It is invariant under the transformation $S \to (V - S)$, as it must be, because any of the two regions of M, in which the loop γ divides it, can be considered as the interior of γ.

3 Two-dimensional gauge theories on the lattice

Here we consider the case of arbitrary gauge group G and arbitrary compact orientable two-dimensional manifold M. We assume that γ divides M into two regions σ_1, σ_2 of genera r_1, r_2 and areas S_1, S_2 respectively. The genus of M is $r = r_1 + r_2$ and its area $V = S_1 + S_2$.

To perform the calculation we introduce a connected graph, or lattice, Λ on the manifold M which is a union of finite sets whose elements are 0-cells s (sites), 1-cells l (links) and 2-cells p (plaquettes). We consider only such lattices that the closed path $\gamma \subset \Lambda$, that is γ is a sequence of links of Λ. We denote the area of the plaquette p as $|p|$. The "lattice spacing" a is defined as the minimum distance such that every plaquette is contained in a circle of the radius a. As a system of loops let us choose $\gamma_p = \partial p$, i.e. loops which are boundaries of the plaquettes, and a set of generators a_i, b_i of the fundamental group of M. The index $i = 1, \ldots, g$ labels the handles of M. Any loop on the lattice can be obtained from the system $\{\gamma_p, a_i, b_i\}$. The loops a_i, b_i allow us to be in any homotopy class while the loops γ_p generate continuum deformations. This system is not independent but subjected to one relation:

$$\prod_{p\in\Lambda} \gamma_p = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \tag{10}$$

from which one can construct an independent system of loops β_j $(j = 1, 2, \ldots, n = \dim \pi_1(\Lambda))$. The product in (10) has to be taken in a certain order which can be shown to exist. Also we would like to note that for the rigorous treatment independent hoops, that are classes of holonomically equivalent loops, should be considered. We will skip this distinction here.

Using the results of [22] (see also [11]) one can give a well-defined meaning to the heuristic measure $\mathcal{D}A$ in Eq. (3) as a measure constructed out of copies of the Haar measure on the group G. This is possible if the action can be written as a cylindrical function on the space $\mathcal{A}/\mathcal{G}$. The standard Yang-Mills action $\mathcal{S}(A)$ is not well defined on such space, hence one has to use some regularized action $\mathcal{S}_\Lambda(A)$ on the lattice Λ, calculate $Z_\Lambda(\gamma)$, which is a lattice analog of (3), and then take the limit $Z(\gamma) = \lim_{a\to 0} Z_\Lambda(\gamma)$.

In this article we consider a class of actions such that

$$e^{-\mathcal{S}_\Lambda(A)} = \prod_{p\in\Lambda} e^{-\mathcal{S}_1(H_{\gamma_p}(A))},$$

where the holonomy $H_\gamma(A)$ is defined by Eq. (1). The function $\mathcal{S}_1$ is a real function over G which satisfies the following conditions: 1) $\mathcal{S}_1(g) = \mathcal{S}_1(g^{-1})$; 2) it has an absolute minimum on the identity element; 3) $\lim_{a\to 0} \mathcal{S}_1(H_p(A))|p|^{-1} = Tr(F_{\mu\nu}(x))^2/2$. An important example is given by the Wilson action: $\mathcal{S}_1(g) = 1 - Re\chi_F(g)/d_F$, where F stands for the fundamental representation.

With such actions the integrand in (3) is a cylindrical function. Let us write the action in terms of the Fourier coefficients:

$$\mu_p(g) \equiv e^{-\mathcal{S}_1(g)} = \sum_R d_R \chi_R(g) B_R(p), \tag{11}$$

where the sum goes over all irreducible representations of G. Any action can be specified through the coefficients B_R. A relevant case is the heat-kernel action given by $B_R(p) = \tilde{B}_R^{|p|}$, $\tilde{B}_R = \exp(-e^2 c_2(R)/2)$,

where $c_2(R)$ is the value of the second Casimir operator. When $G = U(1)$ the heat-kernel action is called Villain action.

Let us denote by g_{a_i} and g_{b_i} the group variables corresponding to the generators a_i and b_i of the homotopy group and by $g_p \equiv g_{\gamma_p}$ the group variables corresponding to the loops γ_p. The functional integral $Z_\Lambda(\gamma)$ is equal to

$$Z_\Lambda(\gamma) = \int_{G^{n+1}} (\prod_i dg_{a_i} dg_{b_i}) \left(\prod_{p\in\Lambda} dg_p \mu_p(g_p) \right) \Delta[\{g_q\}] \frac{1}{d_R} \chi_R \left(\prod_{q\in\gamma} g_q \right). \tag{12}$$

Here dg is the Haar measure on G and $\Delta[\{g_q\}]$ is a factor which imposes the relation between the group variables implied by (10):

$$\Delta[\{g_q\}] = \delta_G(\prod_{p\in\Lambda} g_p, \prod_i g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1}),$$

where $\delta_G(g,h)$ is the Dirac distribution on the group G. A specific decomposition of γ, which appeares in the argument of the character χ_F in (12), is given in terms of the loops located at one of the regions, say, σ_1, namely

$$\gamma = \prod_p^{(1)} \gamma_p [\prod_i^{(1)} a_i b_i a_i^{-1} b_i^{-1}]^{-1}$$

where the superscript (1) means the restriction to σ_1.

Using Eq. (11) and interchanging the summation in (11) with the product over the plaquettes in (12) we obtain that

$$\begin{aligned} Z_\Lambda(\gamma) &= \sum_{\{R_p\}} (\prod_{p\in\Lambda} B_{R_p}(p)) \int_{G^{n+1}} (\prod_i dg_{a_i} dg_{b_i}) \left(\prod_{p\in\Lambda} dg_p \right) (\prod_{p\in\Lambda} d_{R_p} \chi_{R_p}(g_p)) \\ &\times \Delta[\{g_q\}] \frac{1}{d_R} \chi_R \left(\prod_{q\in\gamma} g_q \right). \end{aligned}$$

Here the first sum goes over all possible "colorations" of the surface, i.e. over all possible configurations of the irreducible representations, where each configuration $\{R_p\}$ is a combination of irreducuble representations assigned to each plaquette.

To perform the g_p integration we use a procedure of "lattice reduction". If plaquettes p_1 and p_2 share a link that does not belong to γ (then, they are in the same σ_i component) we change the group variable $g_{p_1} \to g_{p_1} g_{p_2}^{-1}$. The measure of integration is invariant under such changes, g_{p_2} disappears in Δ and χ_F, and we can integrate over g_2 using the properties of invariant group integration. The result is non-zero if both plaquettes are in the same representation, in which case the common link is "erased" and the two plaquettes fuse into one plaquette while the form of the partition function remains the same. Such reductions can be done until we arrive to a lattice consisting only of two plaquettes, p_1 and p_2 with representations R_1 and R_2 respectively. Integration over the variables of the homotopically non-trivial loops gives rise to the factors $d_{R_1}^{1-2r_1}$ and $d_{R_2}^{1-2r_2}$. Finally we obtain that

$$Z_\Lambda(\gamma) = \sum_{R_1,R_2} (\prod_{p\in\sigma_1} B_{R_1}(p)) (\prod_{p\in\sigma_2} B_{R_2}(p)) d_{R_1}^{1-2r_1} d_{R_2}^{1-2r_2} \{R_1, R, R_2\},$$

where

$$\{R_1, R, R_2\} = \int_G dg \chi_{R_1}(g) \chi_{R_2}(g^{-1}) \frac{1}{d_R} \chi_R(g)$$

is the number of times the representation R_2 is contained in $R_1 \otimes R$.

To obtain the continuum limit one should, in general, compute the $B_R(p)$ coefficients and take the limit $|p| \to 0$. This is the case if we work with the Wilson action. If instead we use the heat-kernel action, $Z_\Lambda(\gamma)$ can be written in the form

$$Z_\Lambda(\gamma) = \sum_{R_1,R_2} \tilde{B}_{R_1}^{S_1} \tilde{B}_{R_2}^{S_2} d_{R_1}^{1-2r_1} d_{R_2}^{1-2r_2} \{R_1, R, R_2\}$$

which is lattice independent, so it gives already the continuum limit. The final expression for the continuum limit of the vacuum expectation value of the Wilson loop functional is

$$< T_\gamma >= \frac{\sum_{R_1,R_2} \tilde{B}^{S_1}_{R_1} \tilde{B}^{S_2}_{R_2} d^{1-2r_1}_{R_1} d^{1-2r_2}_{R_2} \{R_1, R, R_2\}}{d_R \sum_{R_3} \tilde{B}^{V}_{R_3} d^{2-2r}_{R_3}}. \tag{13}$$

For abelian groups $d_R = 1$. In this case the topology of the surface is irrelevant. If $G = U(1)$ the irreducible representations are labelled by integers $n \in Z$. Since $B_n = e^{-e^2 n^2/2}$, taking $R = \nu$ we get

$$< T_\gamma >= \frac{\sum_{n=-\infty}^{\infty} B^{S_1}_n B^{S_2}_{n+\nu}}{\sum_{n=-\infty}^{\infty} B^{V}_n} = \frac{\sum_{n=-\infty}^{\infty} e^{-S_1 e^2 n^2/2} e^{-S_2 e^2 (n+\nu)^2/2}}{\sum_{n=-\infty}^{\infty} e^{-V e^2 n^2/2}}.$$

This coincides with (9) after the identifications $S_1 = S$ and $S_2 = V - S$.

We would like to make a few comments. The result (13) is invariant under the transformation $S \leftrightarrow V - S$. Although we have considered only compact surfaces, the same techniques can be applied in the non-compact case. The important difference is that the infinite component at one side of γ appears to be in the trivial representation. This case can also be analized as the limit $V \to \infty$ of a compact case. Performing it in (13) we see that the Wilson loop expectation value for $\mathbf{R^2}$ or any non-compact surface has the typical area law behaviour $< T_\gamma > \propto \exp\left(-e^2 c_2(R) S/2\right)$, where S is the area of the compact part enclosed by γ. For finite V similar behaviour is obtained in the strong coupling limit $e^2 V \gg 1$ in which case only the region with the smaller area contributes. In the weak coupling limit $e^2 V \ll 1$ for the abelian case

$$< T_\gamma > \sim e^{-e^2 (S - S^2/V) \nu^2/2}.$$

4 Wilson loop variables for invariant connections.

In Sect. 2, while calculating the vacuum expectation value of the Wilson loop functional in the abelian case, we analyzed the contribution of the monopoles. In the present section we will study this quantity for a special class of connections, called invariant connections, for $M = S^2$ and $G = SU(2)$. The result of the calculation will be compared with the exact formula derived above. This will allow us to understand how much of the information is captured by the invariant connections. Our interest in this class of connections is motivated by the fact that the analogous calculation can be carried out in Yang-Mills theory in any dimension provided, of course, that non-trivial invariant connections exist.

Let us first discuss the definition of the invariant connections and explain how they can be constructed. After that these results will be applied to the particular example with $M = S^2$ and $G = SU(2)$. Assume that there is a group K which acts on the space-time M and its action can be lifted to the fibre bundle $P(M, G)$. We will call K symmetry group. A connection on P is said to be invariant with respect to transformations L_k of K if its connection 1-form w satisfies $L^*_k w = w$ for all $k \in K$. This class of connections includes many known monopole and instanton configurations and they were intensively used for the coset space dimensional reduction of gauge theories (see [23]). Gauge potentials corresponding to invariant connections are called symmetric potentials and were introduced in [24]. The condition of invariance implies that for any $k \in K$ there exists a gauge tranformation $g_k(x) \in G$ such that the gauge potential A_μ, corresponding to the invariant connection form w, has the property

$$(O_k A)_\mu = g_k(x)^{-1} A_\mu(x) g_k(x) + \frac{1}{e} g_k(x)^{-1} \partial_\mu g_k(x), \tag{14}$$

where the l.h.s. is the standard change of the field under the space-time transformation O_k on M (which is the projection of L_k to M). This formula tells that the symmetric potential is invariant under transformations of K up to a gauge transformation.

Here we consider the case when K acts transitively on M (see Ref. [13] for general theory and Ref. [25] for review). Then M is a coset space K/H, where H is a subgroup of K, called the isotropy group, and K acts on K/H in the canonical way. The 1-form $A = A_\mu dx^\mu$, which describes a symmetric potential satisfying (14) and is a pull - back of an invariant form with respect to a (local) section of $P(M, G)$, can be constructed in the following way. Let $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{H}$ be the Lie algebras of the groups G, K and H

respectively. If H is a closed compact subgroup of K, the case we have in mind, then there exists the decomposition

$$\mathcal{K} = \mathcal{H} + \mathcal{M} \tag{15}$$

with $ad(\mathcal{H})\mathcal{M} = \mathcal{M}$, where ad denotes the adjoint action of the algebra on itself, i.e. $ad(h)(u) = [h, u]$, $h, u \in \mathcal{G}$. Let θ be the canonical left-invariant 1-form on the Lie group K with values in $\mathcal{K}$, and $\bar{\theta}$ is its pull-back to $M = K/H$. It can be calculated as $\bar{\theta} = k(x)^{-1}dk(x)$, where $k(x) \in K$ is a local representative of the class $x \in K/H$. We decompose the 1-form $\bar{\theta}$ into the $\mathcal{H}$- and $\mathcal{M}$-components in accordance with (15): $\bar{\theta} = \bar{\theta}_{\mathcal{H}} + \bar{\theta}_{\mathcal{M}}$. It can be shown that the invariant connections are given by

$$A = \frac{1}{e}\left(\tau(\bar{\theta}_{\mathcal{H}}) + \phi(\bar{\theta}_{\mathcal{M}})\right), \tag{16}$$

where τ is the homomorphism of algebras induced by a group homomorphism $H \to G$ and ϕ is a mapping $\phi : \mathcal{M} \to \mathcal{G}$ satisfying the equivariant condition

$$\phi\left(ad(h)u\right) = ad(\tau(h))\phi(u), \quad h \in \mathcal{H}, \quad u \in \mathcal{M}. \tag{17}$$

This condition can be viewed as the intertwining condition between representations of $\mathcal{H}$ in $\mathcal{M}$ and $\mathcal{G}$. An effective technique for solving the constraint (17) was developed in Refs. [27] (see also [25]).

As a concrete example we will consider the Yang-Mills theory on the two-dimensional sphere S^2 with the gauge group $G = SU(2)$. The sphere is realized as a coset space $S^2 = SU(2)/U(1)$. Let us construct first the 1-forms $\bar{\theta}_{\mathcal{H}}$ and $\bar{\theta}_{\mathcal{M}}$ which appear in Eq. (16). As usual, we cover the manifold $M = S^2$ with two charts U_1 and U_2, the northern and southern hemispheres, so that $U_1 \cup U_2 \cong S^2$ and $U_1 \cap U_2 = U_{12} \cong S^1$ is the equator. More precisely, if ϑ and φ are two angles parametrizing points of the sphere, then $U_1 = \{0 \le \vartheta \le \pi/2, 0 \le \varphi < 2\pi\}$, $U_2 = \{\pi/2 \le \vartheta \le \pi, 0 \le \varphi < 2\pi\}$. We take the generators of $K = SU(2)$ to be $Q_i = i\tau_i/2$ $(i = 1, 2, 3)$, where τ_i are the Pauli matrices. The subgroup $H = U(1)$ is generated by Q_3 and the vector space $\mathcal{M}$ is spanned by Q_1 and Q_2. Let us consider the decomposition of the algebra $\mathcal{K}$ in the case under consideration. Denote by $e_{\pm\alpha} = \tau_{\pm} = (\tau_1 \pm i\tau_2)/2$ the root vectors and by $h_\alpha = \tau_3$ the corresponding Cartan element of this algebra.

We choose local representatives $k^{(j)}(\vartheta, \varphi) \in K = SU(2)$, $j = 1, 2$ of the points of the coset space $S^2 = SU(2)/U(1)$ as follows:

$$k^{(1)}(\vartheta, \varphi) = e^{\varphi Q_3} e^{\vartheta Q_2} e^{-\varphi Q_3} \quad \text{and} \quad k^{(2)}(\vartheta, \varphi) = e^{-\varphi Q_3} e^{-(\pi - \vartheta) Q_2} e^{\varphi Q_3}.$$

By straightforward computation one obtains the forms $\bar{\theta}_{\mathcal{H}}$ and $\bar{\theta}_{\mathcal{M}}$ as

$$\bar{\theta}^{(i)} = \left(k^{(i)}\right)^{-1} dk^{(i)} = \bar{\theta}^{(i)}_{\mathcal{H}} + \bar{\theta}^{(i)}_{\mathcal{M}}.$$

According to the general formula (16) the invariant gauge connection depends on the gauge group and the embedding $\tau(H) \subset G$. The decomposition of the vector space $\mathcal{M}$ into irreducible invariant subspaces of $\mathcal{H}$ corresponds to the following decomposition of representations:

$$\underline{2} \to (2) + (-2), \tag{18}$$

where in the r.h.s. we indicated the eigenvalues of $ad(h_\alpha)$, and the reducible representation carried by the space $\mathcal{M}$ is indicated by its dimension in the l.h.s.

Now let E_α, $E_{-\alpha}$ and H_α be the root vectors and the Cartan element of the algebra A_1, which appears as complexification of $\mathcal{G} = su(2)$. We assume that they are given by the same combinations of the Pauli matrices as the corresponding elements of complexified $\mathcal{K}$ described above. Define the group homomorphism $\tau : H = U(1) \to G = SU(2)$ by the expression $\tau\left(\exp(i\tau_3\alpha_3)\right) = \exp(i\kappa\tau_3\alpha_3) = \cos(\kappa\alpha_3) + i\tau_3 \sin(\kappa\alpha_3)$. It is easy to check that this definition is consistent if κ is an integer. Thus the corresponding homomorphism of algebras is again labelled by $n \in Z$ and is given by $\tau_n(h_\alpha) = nH_\alpha$. The three-dimensional space $\mathcal{G}$ of the adjoint representation of A_1 decomposes into three 1-dimensional irreducible invariant subspaces of $\tau_n(\mathcal{H})$:

$$\underline{3} \to (0) + (2n) + (-2n) \tag{19}$$

(again we indicate the eigenvalues of $ad(\tau(H_\alpha))$). Compare the decompositions (18) and (19). For $n \neq \pm 1$ there are no equivalent representations in the decompositions of $\mathcal{M}$ and $\mathcal{G}$, and the intertwining operator $\varphi : \mathcal{M} \to \mathcal{G}$ is zero. It also turns out to be zero for $n = 0$. In accordance with (16) for $n \neq \pm 1$

$$A_n^{(1)} = \frac{in}{2e}\tau_3(1 - \cos\vartheta)d\varphi, \quad A_n^{(2)} = -\frac{in}{2e}\tau_3(1 + \cos\vartheta)d\varphi. \tag{20}$$

If $n = 1$ or $n = -1$ the results are more interesting. Let us consider the case $n = 1$ in detail. Comparing again (18) and (19) we see that there are pairs of representations with the same eigenvalues and, therefore, the operator ϕ is non-trivial. It is determined by its action on basis elements of $\mathcal{M}$: $\phi(e_\alpha) = f_1 E_\alpha$, $\phi(e_{-\alpha}) = f_2 E_{-\alpha}$, where f_1, f_2 are complex numbers. The fact that the initial groups and algebras are compact implies a reality condition [25] which tells that $f_1 = f_2^*$. So, the operator ϕ and the invariant connection are parametrized by one complex parameter f_1 (we will suppress its index from now on). Using the formulas above we obtain that

$$A^{(1)} = \frac{i}{2e}\begin{pmatrix} (1 - \cos\vartheta)d\varphi & fe^{-i\varphi}(id\vartheta + \sin\vartheta d\varphi) \\ f^* e^{i\varphi}(-id\vartheta + \sin\vartheta d\varphi) & -(1 - \cos\vartheta)d\varphi \end{pmatrix}, \tag{21}$$

$$A^{(2)} = \frac{i}{2e}\begin{pmatrix} -(1 + \cos\vartheta)d\varphi & fe^{i\varphi}(id\vartheta + \sin\vartheta d\varphi) \\ f^* e^{-i\varphi}(-id\vartheta + \sin\vartheta d\varphi) & (1 + \cos\vartheta)d\varphi \end{pmatrix}. \tag{22}$$

The curvature form F is given by

$$F = dA + \frac{e}{2}[A, A] = -\frac{i}{2e}\tau_3\left(|f|^2 - 1\right)\sin\vartheta d\vartheta \wedge d\varphi, \tag{23}$$

and the action for such configuration is equal to

$$\mathcal{S}_{inv} = \frac{2\pi}{e^2}\frac{1}{R^2}\left(|f|^2 - 1\right)^2, \tag{24}$$

where R is the radius of the sphere. Due to the invariance property any extremum of the action found withing the subspace of invariant connections is also an extremum in the space of all connections [25]. From Eq. (24) we see that there are two extrema: the trivial one with $f = 0$ and the non-trivial one with $|f| = 1$. The trivial extremum was found in Ref. [28] as a spontaneous compactification solution in six-dimensional Kaluza-Klein theory.

It turns out that the potentials (20), (21) and (22) are related to known non-abelian monopole solutions (see, for example, [29]). For $n \neq \pm 1$ and for $n = \pm 1$ with $f = 0$ these expressions coincide with the monopole solutions with the even monopole charge $m = 2n$. In fact the solution with the charge $m > 0$ can be transformed to the solution with the charge $(-m)$ by the gauge transformation $A \to S^{-1}AS$ with the constant matrix $S = -i\tau_1$. All these monopoles live in the trivial principal fibre bundle $P(S^2, SU(2))$, which is the only bundle with such structure group. The latter result also follows from our discussion in Sect. 2. Indeed, in the case under consideration $\mathcal{B}_{SU(2)}(S^2) = H^2(S^2, \pi_1(SU(2))) = 0$ since $SU(2)$ is simply connected. Thus, all the monopoles can be described by a unique function on the whole sphere. This is indeed the case. Namely, there exist gauge transformations, different for the northern and southern patches, so that the tranformed potentials coincide. Let us show this for the case $n = 1$. In fact we will construct such transformations for the whole family of the invariant connections (21), (22). The matrices giving these gauge transformations are

$$U_1 = \begin{pmatrix} \cos\frac{\vartheta}{2} & e^{-i\varphi}\sin\frac{\vartheta}{2} \\ -e^{i\varphi}\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix}, \quad U_2 = \begin{pmatrix} \cos\frac{\vartheta}{2}e^{i\varphi} & \sin\frac{\vartheta}{2} \\ -\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2}e^{-i\varphi} \end{pmatrix}.$$

By calculating $A^{(i)'} = U_i^{-1}A^{(i)}U_i + g^{-1}U_i^{-1}dU_i$ for $i = 1$ and $i = 2$ one can easily check that the transformed fields are equal to

$$A^{(1)'} = A^{(2)'} = \frac{i}{2e}\left(\tau_+ c_+ + \tau_- c_- + \tau_3 c_3\right) \tag{25}$$

with

$$\begin{aligned} c_+ &= c_-^* = e^{-i\varphi}\left\{\left[-\sin\vartheta\cos\vartheta + \left(f\cos^2\frac{\vartheta}{2} - f^*\sin^2\frac{\vartheta}{2}\right)\sin\vartheta\right]d\varphi\right. \\ &+ \left. i(-1 + f\cos^2\frac{\vartheta}{2} + f^*\sin^2\frac{\vartheta}{2})d\vartheta\right\} \\ c_3 &= \left(1 - \frac{f+f^*}{2}\right)\sin^2\vartheta d\varphi - i\frac{f-f^*}{2}\sin\vartheta d\vartheta. \end{aligned}$$

Note that in the general expressions (21), (22) and (25) the phase of the complex parameter f can be rotated by residual gauge transformations which form the group $U(1)$.

For $f = 0$ this formula gives the known expression for the $m = 2$ $SU(2)$-monopole [29]:

$$A^{(1)} = A^{(2)} = \frac{i}{4e}\begin{pmatrix} (1-\cos 2\vartheta)d\varphi & e^{-i\varphi}(-2id\vartheta - \sin 2\vartheta d\varphi) \\ e^{i\varphi}(2id\vartheta - \sin 2\vartheta d\varphi) & -(1-\cos 2\vartheta)d\varphi \end{pmatrix}. \tag{26}$$

One can easily see that the forms (25) and (26) do not have singularities on the sphere.

For $f = f^* = 1$ the form (25) vanishes. This shows that this configuration, which is also an extremum of the action, describes the trivial $SU(2)$-monopole with $m = 0$. Note that in the original form the potentials (21) and (22) do not seem to be trivial. Of course, one can check that they are pure gauge configurations and correspond to the flat connection. Vanishing of the gauge field (23) for this value of f confirms this.

A similar situation occurs for $n = -1$. Again, there exists a family of invariant connections parametrized by a complex parameter, say h, analogous to f. The action possesses two extrema. One, with $h = 0$, describes the $SU(2)$-monopole solution with $m = -2$ and the other extremum, with $|h| = 1$, describes the trivial monopole with $m = 0$. However, this family of invariant connections is gauge equivalent to the family for $n = +1$ and can be obtained from (21), (22) by the gauge transformation with the constant matrix $S = -i\tau_1$ and the identification $h = f$.

The picture we obtained is the following. Various homomorphisms $\tau_n : H \to G$ give rise to various invariant connections describing $SU(2)$-monopole solutions. All monoples are reproduced in this way. Moreover there is a continuous family of invariant gauge connections, parametrized by one complex parameter, f, which passes through the points corresponding to $SU(2)$-monopoles with $m = 0$ ($n = 0$), $m = 2$ ($n = 1$) and $m = -2$ ($n = -1$) in the space of all connections of the theory. Classes of gauge equivalent invariant connections are labelled by values of $|f|$. Thus, $|f| = 0$ corresponds to the class of the $m = 2$ monopole (the $m = -2$ monopole is in the same class, as we already explained above), $|f| = 1$ corresponds to the class of the $m = 0$ monopole.

Let us calculate now the contribution of the invariant connections to the vacuum expectation value of the Wilson loop functional. For this we choose a loop $\gamma(\vartheta_0)$ on S^2 given by

$$\gamma(\vartheta_0) = \{(\vartheta_0, \varphi'), \vartheta_0 = \text{const}, 0 \le \varphi' < 2\pi\}. \tag{27}$$

This loop is parallel to the equator, is labelled by the angle ϑ_0 and is parametrized by the polar angle φ'. Here for definiteness we consider the case when the loop lies in the northern hemisphere, i.e. $0 \le \vartheta_0 \le \pi/2$.

We notice that the group element

$$U(\varphi) = \mathcal{P}\exp\left[ie\int_{\eta(\varphi;\vartheta_0)} A_\phi^{(1)} d\varphi'\right],$$

where $\eta(\varphi;\vartheta_0)$ is the same path as (27) but with φ' changing from 0 to φ, satisfies the matrix equation

$$\frac{dU(\varphi)}{d\varphi} = ieU(\varphi)A_\varphi^{(1)}(\vartheta_0,\varphi). \tag{28}$$

The holonomy $H_{\gamma(\vartheta_0)}(A^{(1)}) = U(2\pi)$. It turns out that Eq. (28) can be solved and the traced holonomy is equal to

$$T_{\gamma(\vartheta_0)}(A^{(1)}) = -\frac{1}{2}\cos\left(\pi\sqrt{1 + \frac{4S}{V}\left(1 - \frac{S}{V}\right)(|f|^2 - 1)}\right), \tag{29}$$

where $S = 2\pi R^2(1-\cos\vartheta_0)$ is the area of the surface surrounded by the loop $\gamma(\vartheta_0)$ and $V = 4\pi R^2$ is the total area of the two-dimensional sphere of the radius R. Note that for $|f| = 1$ $T_{\gamma(\vartheta_0)} = 1$ as it should be for a flat connection. The expression (29) is invariant under the transformation $S \to (V - S)$.

Finally we evaluate the quantity $< T_{\gamma(\vartheta_0)} >_{inv} = Z_{inv}(\gamma(\vartheta_0))/Z_{inv}(0)$, characterizing the contribution of the invariant connections to the vacuum expectation value of the Wilson loop functional, where

$$Z_{inv}(\gamma(\vartheta_0)) = -\int df\ df^* e^{-\frac{8\pi^2}{g^2 V}\left(|f|^2-1\right)^2} \cos\left(\pi\sqrt{1+\frac{4S}{V}\left(1-\frac{S}{V}\right)\left(|f|^2-1\right)}\right). \tag{30}$$

This integral is an analog of the functional integral (4). It mimics a "path-integral quantization" of the gauge model where the configuration space (the space of connections) is finite-dimensional and consists of $SU(2)$ - invariant connections. The integral takes into account the contribution of the monopoles with $m = 0, \pm 2$ and fluctuations around them along the *invariant* direction. The action on the invariant connection in the formula above is given by Eq. (24).

Of course, the complete contribution of the invariant connections, given by Eqs. (21), (22) or Eq. (25) to the true functional integral (4) is different because it includes also contributions of all fluctuations around them. In the next section we are going to discuss which part of the true vacuum expectation value $< T_{\gamma(\vartheta_0)} >$ is captured by $< T_{\gamma(\vartheta_0)} >_{inv}$.

We would like to make a few remarks. In Ref. [30] it was argued that the Faddev-Popov determinant on invariant connections turns out to be zero, hence the contribution of such connections to the functional integral vanishes. This issue was analyzed in [31] in a different context, namely the authors considered similar "quantization" for $SU(2)$-invariant connections in $(3+1)$ - dimensional Ashtekar's gravity. They found that zeros of the Faddev-Popov determinant, which are responsible for the vanishing of the determinant, are cancelled by the contribution of the delta functions of constraints and the path integral measure is regular on invariant connections.

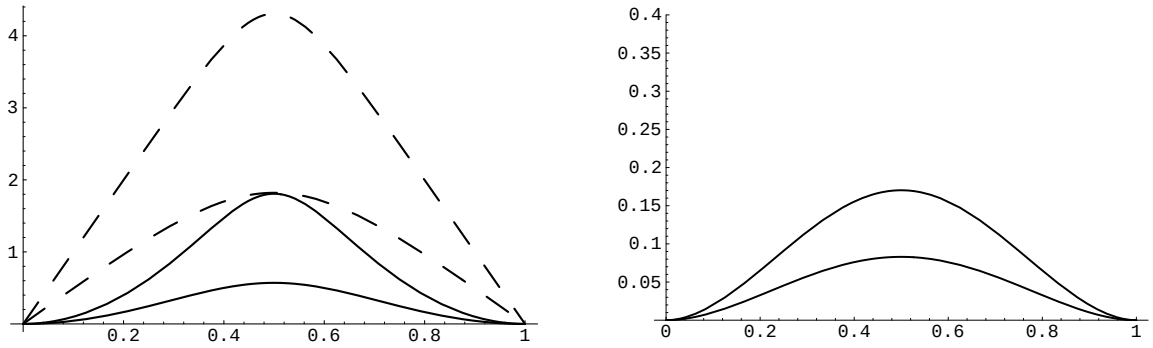
5 Discussion of the result

In Sect. 2 and Sect. 3 we calculated the vacuum expectation value of the Wilson loop functional. Let us analyze the quantity $E(g^2V, S/V) \equiv -\log < T_{\gamma(\vartheta_0)} >$ which in a theory with fermions characterizes the potential of the interaction. Recall that the closed path $\gamma(\vartheta_0)$ on S^2 was defined by Eq. (27), S is the area of the region surrounded by this loop and its ratio to the total area of the surface is given by $S/V = (1-\cos\vartheta_0)/2$. $E(g^2V, S/V)$ as a function of S/V for two values of e^2V in the *abelian* case is given by Eq. (9) and is presented in Fig. 1a. In the same plot we also show the contribution $E_{mon}(g^2V, S/V)$ of the *abelian* monopoles calculated from (8). We see from Fig. 1a (also from the exact formula (8)) that the dependence of $E_{mon}(g^2V, S/V)$ on S/V at small S/V is quadratic almost till $S/V = 0.5$, where the curve reaches its maximum. This, being combined with the contribution of the fluctuations, gives the linear dependence (the area law) almost for all S/V in the interval $0 < S/V < 0.5$. The area law for the Wilson loop in the pure Yang-Mills theory is considered as an indication of the regime of confinement in the corresponding gauge theory with quarks. The behaviour of $E(g^2V, S/V)$ for the *non-abelian* theory with $G = SU(2)$ is qualitatively the same. In this case contributions of fluctuations around a monopole depend on the monopole solution, so that the complete $Z(\gamma)$ cannot be written in the factorized form as in the abelian case. However, the sum of the terms $\exp(-\mathcal{S}_{mon})T_\gamma(A_{mon})$ over all monopoles gives rise to qualitatively the same behaviour as $E_{mon}(g^2V, S/V)$ in the abelian case.

As we have pointed out the linear behaviour of the total $E(g^2V, S/V)$ is a manifestation of the area law for "not very large" S/V, i.e. when the area surrounded by the loop γ is far below the half of the total area of the compact space-time M. In this respect the quadratic behaviour of the monopole contributions $E_{mon}(g^2V, S/V)$ seems to be an important feature which gives rise to the linear dependence of the complete function.

The plot of the energy $E_{inv}(g^2V, S/V) \equiv -\log < T_{\gamma(\vartheta_0)} >_{inv}$ as a function of S/V is given by Eq. (30) and is shown in Fig. 1b. Comparing it with Fig. 1a we see qualitative similarity in the contribution of invariant connections and monopoles, and we want to attract attention to this feature. It can be supposed that the quadratic behaviour of the contribution of invariant connections, similar to that of the monopoles, also leads to the linear dependence of the complete $E(g^2V, S/V)$ in the $SU(2)$ case and,

thus, serves as an idicator of the area law behaviour of Wilson loop variables in the theory. If this is also true in higher dimensions, one could study just the contribution of invariant connections and from the form of the behaviour of the function E_{inv} for invariant connections get some hints about the type of the behaviour of the function E. We are going to study this possibility in a future publication.



(a) (b)

Figure 1. (a) $E(g^2V, S/V)$ (dashed line) and $E_{mon}(g^2V, S/V)$ (solid line) as functions of S/V for $g^2V = 10$ (lower lines) and $g^2V = 20$ (upper lines) in the abelian gauge theory.

(b) Contribution of the invariant connections $E_{inv}(g^2V, S/V)$ as a function of S/V for $g^2V = 10$ (lower line) and $g^2V = 20$ (upper line) in the $SU(2)$ Yang-Mills theory.

Acknowledgments

We would like to thank Jose Mourão and Sebastià Xambó for useful discussions and valuable remarks. J.M.A. acknowledges the financial support from CICYT (project PB94-1196). Y.K. acknowledges the financial support from CIRIT (Generalitat de Catalunya), RFFR (grant 96-0216413-a), INTAS (grant INTAS-93-1630.EXT) and JNICT (grant JNICT/CERN/S/FAE/1111/96).

References

[1] E. Witten, *Comm. Math. Phys.* **141** (1991) 153; *J. Geom. Phys.* **9** (1992) 303.

[2] M. Douglas and V. Kazakov, *Phys. Lett.* **B319** (1993) 219.

[3] G. Gross and W. Taylor, *Nucl. Phys.* **B400** (1993) 161; *ibid* **B403** (1993) 395.

[4] R. Dijkgraaf, E. Verlinde and H. Verlinde, hep-th/9709107.

[5] M.F. Atiyah and R. Bott, *Philos. Trans. R. Soc. London* **A308** (1982) 523; *Topology* **23** (1984) 1.

[6] D.S. Fine, *Comm. Math. Phys.* **134** (1990) 273; *ibid* **140** (1991) 321.

[7] M. Blau and G. Thompson, "Lectures on 2d Gauge Theories" (hep-th/9310144).

[8] G. Moore, "2D Yang-Mills Theory and Topological Field Theory" (hep-th/9409044).

[9] A. Migdal, *Zh. Eksp. Teor. Fiz.* **69** (1975) 810 (*Sov. Ph. JETP* **42** (1975) 413); B. Rusakov, *Mod. Phys. Lett.* **A5** (1990) 693; D.V. Boulatov, *Mod. Phys. Lett.* **A9** (1994) 365.

[10] M. Blau and G. Thompson, *Int. J. Mod. Phys.* **A7** (1992) 3781.

[11] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão and T. Thiemann, *J.Math.Phys.* **38** (1997) 5453.

[12] G. Grignani, G.W. Semenoff and P. Sodano, hep-th/9503109; G. Grignani, L. Paniak, G.W. Semenoff and P. Sodano *Ann. Phys.* **260** (1997) 275; G. Mitreuter, J.M. Pawlowski and A. Wipf, hep-th/9611105.

[13] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Intersci., N.Y. (1963).

[14] A. Trautman, *Rep. Math. Phys.* **1** (1970) 29; Y.M. Cho, *J. Math. Phys.* **16** (1975) 2029; M. Daniel and C.M. Viallet, *Rev. Mod. Phys.* **52** (1980) 175; T. Eguchi, P.B. Gilkey and A.J. Hanson, *Phys. Rep.* **66** (1980) 213; C. Nash and S. Sen, Topology and Geometry for Physicists, Academic Press, London (1983).

[15] K. Wilson, *Phys. Rev.* **D10** (1974) 2445.

[16] R. Giles, *Phys. Rev.* **D8** (1981) 2160.

[17] L. Schulman, *Phys. Rev.* **176** (1968) 1558; *J. Math. Phys.* **12** (1971) 304; M.G.G. Laidlaw and C.M. DeWitt, *Phys. Rev.* **D3** (1971) 1375; M.A. Asorey, J.G. Esteve and A.F. Pacheco, *Phys. Rev.* **D27** (1983) 1852.

[18] S.J. Avis and C.J. Isham, in: "Recent Developments in Gravitation. Cargèse 1978.", Eds. M. Lévy and S. Deser (plenum Press, NY, 1979), p. 347.

[19] J.M. Aroca and Yu.A. Kubyshin, in preparation.

[20] E.H. Spanier, Algebraic Topology, McGraw Hill, N.Y. (1966).

[21] B. Felsager, Geometry, Particles and Fields, Odense University Press (1981).

[22] A. Ashtekar and J. Lewandowski, *J. Math. Phys.* **36** (1995) 2170.

[23] N.S. Manton, *Nucl. Phys.* **B158** (1979) 141; P. Forgacs and N.S. Manton, *Commun. Math. Phys.* **72** (1980) 15.

[24] V.N. Romanov, A.S. Schwarz and Yu.S. Tyupkin, *Nucl. Phys.* **B130** (1977) 209; A.S. Schwarz, *Commun. Math. Phys.* **56** (1977) 79.

[25] Yu.A. Kubyshin, J.M. Mourão, G. Rudolph and I.P. Volobujev, Dimensional reduction of gauge theories, spontaneous compactification and model building. Lecture Notes in Physics, vol. 349. Springer-Verlag, Berlin (1989).

[26] V. Hussin, J. Negro and M.A. del Olmo, *Ann. Phys.* **231** (1994) 211; Yu.A. Kubyshin, O. Richter and G. Rudolph, *J. Math. Phys.* **34** (1993) 5268; Yu.A. Kubyshin, V.O. Malyshenko and D. Marín Ricoy, *J. Math. Phys.* **35** (1994) 310.

[27] Yu.A. Kubyshin, J.M. Mourão and I.P. Volobujev, *Int. J. Mod. Phys.* **A4** (1989) 151; *Nucl. Phys.* **B322** (1989) 531; Yu.A. Kubyshin, *J. Math. Phys.* **32** (1991) 2200.

[28] S. Randjbar-Daemi and R. Percacci, *Phys. Lett.* **117B** (1982) 41.

[29] Chan Hong-Mo and Tsou Sheung Tsun, Some elementary gauge theory concepts. World Scientific, Singapore (1993).

[30] E. Mottola, *J. Math. Phys.* **36** (1995) 2470.

[31] S. Aleksandrov, I. Grigentch and D. Vassilevich, gr-qc/9705080.