Corrections of order $(Z\alpha)^6\frac{m_e^2}{m_\mu}$ in the muonium fine structure, induced by three photon exchange diagrams

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Abstract

In the framework of the quasipotential method we calculate the contributions of the kind $\alpha^6\ln\alpha$, α^6 of three-photon exchange diagrams to the energy spectrum of the muonium n^3S_1 state in the leading order on m_e/m_μ. The analytical expression of the obtained correction for arbitrary principal quantum number is equal to $(Z\alpha)^6(m_e/m_\mu)m_e(6\ln(2)-35/48)/n^3$.

The investigation of muonium and positronium fine structure represents one of the basic tests of quantum electrodynamics, which is sensitive to radiative corrections of higher order on α [1]. Many papers [2, 3, 4, 5] are devoted to the calculation of different contributions to the fine structure of hydrogen-like atom energy levels. The interest to this problem remains unchanged [6, 7, 8]. The progress, achieved in the last years during the calculation of logarithmic contributions of order $\alpha^6\ln\alpha$ in the positronium fine structure intervals $(2^3S_1\div 1^3S_1, 2^3S_1\div 2^3P_J)$ [9, 10, 11], doesn't abolish the necessity of calculation of higher order corrections $O(\alpha^6)$ [12]. The development of experimental methods, based on Doppler-free two-photon spectroscopy, allows the "large" structure intervals to be measured for the muonium and the positronium [13, 14, 15].

$$\Delta E^{exp.}_{Ps}(2^3S_1\div 1^3S_1)=\begin{cases}1233607218,9\pm 10,7\ MHz\\ 1233607216,4\pm 3,2\ MHz\end{cases}\tag{1}$$

$$\Delta E^{exp.}_{Mu}(2^3S_1\div 1^3S_1)=2455527936\pm 120\pm 140\ MHz.\tag{2}$$

The increase of the experimental accuracy of muonium fine structure interval measurements (just as for positronium), planned in the near future, makes very actual the calculation of radiative corrections of higher order on α. In this paper we have performed studies of the contributions of order $O(\alpha^6\ln\alpha), O(\alpha^6)$ from three-photon exchange diagrams to the muonium fine structure. The contribution of these diagrams to the muonium hyperfine structure was obtained in [16]. Our calculations are based on the Schrodinger-type local quasipotential equation [17]

$$\left(\frac{b^2}{2\mu_R}-\frac{\vec p^2}{2\mu_R}\right)\psi_M(\vec p)=\int\frac{d\vec q}{(2\pi)^3}V(\vec p,\vec q,M)\psi_M(\vec q),\tag{3}$$

where $b^2=E_1^2-m_1^2=E_2^2-m_2^2$, $\mu_R=E_1E_2/M$ is relativistic reduced mass, $M=E_1+E_2$ is the bound state mass, m_1, m_2 are the masses of the electron and the muon. As an initial approximation of quasipotential $V(\vec p,\vec q,M)$ for the bound state system $(e^-\mu^+)$ we choose the ordinary Coulomb potential. On the basis of equation (3) in [18] we have obtained some relativistic corrections $m\alpha^6$ in the positronium fine structure from the one-photon, two-photon interaction and the second order perturbation theory. There are six diagrams, shown in Fig.1, which determine the three-photon exchange interaction in the muonium.

Let consider the first diagram in Fig.1. The corresponding amplitude already has the factor α^6, which appears due to electromagnetic vertices and the Coulomb wave function. So, in the first stage of our calculations we have neglected by the electron and muon vector momentum of relative motion in the initial and final states, taking into account that the necessary accuracy is already achieved. Then an amplitude corresponding to the first diagram in Fig.1 takes the form:

$$T_1^{3\gamma}=-\frac{(Z\alpha)^3}{4\pi^5}\int d^4p\int d^4p'\frac{<\gamma_1^\lambda(\hat q_1-\hat p'+m_1)\gamma_1^\nu(\hat p_1-\hat p+m_1)\gamma_1^\mu>}{(p^2-w^2+i\epsilon)(p'^2-w^2+i\epsilon)[(p-p')^2+i\epsilon]}\tag{4}$$

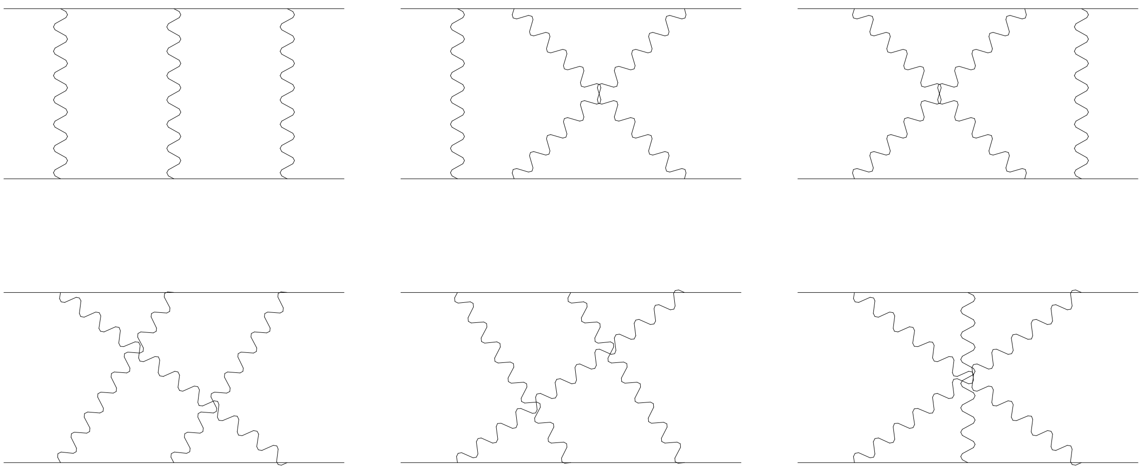

Figure 1: **Feynman diagrams of three-photon exchange interaction in muonium**

$$\frac{<\gamma_2^\mu(\hat p_2+\hat p+m_2)\gamma_2^\nu(\hat q_2+\hat p'+m_2)\gamma_2^\lambda>}{D_e(p)D_e(p')D_\mu(-p)D_\mu(-p')},$$

where $D_{e,\mu}(p)$ are the denominators of electron and muon propagators:

$$D(\pm p)=p^2-w^2\pm 2mp^0+i\epsilon,\quad w^2=-b^2, \tag{5}$$

and the corner brackets designate the averaging on Dirac bispinors; p_1,p_2 are the four-momenta of particles in the initial state; q_1,q_2 are the particle four-momentum in the final state. As usually, the factor $Z\alpha$ emphasizes exchanging character of photon interaction between particles. The exchange photon propagators were taken in the Feynman covariant gauge. As it known, use of the Coulomb gauge is the most natural for exchange photons, because the Coulomb interaction dominates in the system $(e^-\mu^+)$. Nevertheless, the equivalence of Coulomb and Feynman gauges in the scattering approximation for calculations of three-photon diagrams was shown in [16]. To construct the quasipotential of the system $(e^-\mu^+)$ with L=0 and J=1, that corresponds to $T_1^{3\gamma}$, let introduce the projector operator for initial and final states of the following kind [18, 19]:

$$\hat\pi=\frac{1}{2\sqrt2}\frac{\hat P+M}{M}\hat\varepsilon, \tag{6}$$

where $P=p_1+p_2=q_1+q_2$ is full four-momentum of the system, and ε^μ is the vector of $^3S_1$ muonium polarization. Projecting the particles on $^3S_1$- state by means of (6), we avoid cumbersome matrix multiplication in the bispinor averages and immediately pass on to calculation of the total trace in (4). As a result, the quasipotential relevant to the first diagram may be written in the form:

$$V_1^{3\gamma}=-\frac{(Z\alpha)^3}{\pi^5}\int d^4p\int d^4p'\frac{F_1(p,p')}{D_\gamma(p)D_\gamma(p')D_\gamma(p-p')D_e(p)D_e(p')D_\mu(-p)D_\mu(-p')}, \tag{7}$$

$D_\gamma(p)=p^2-w^2+i\epsilon,$

$$F_1(p,p')=f_{12}(p,p')m_2^2+\frac13 f_{11}m_2,\quad f_{12}=pp'-4m_1^2-2m_1p_0-2m_1p_0'-2p_0p_0', \tag{8}$$

$$f_{11}(p,p')=2m_1p'^2+p_0p'^2+10m_1pp'+2p_0pp'+2p_0'pp'+$$
$$2m_1p^2+p_0'p^2+6m_1^2p_0+6m_1^2p_0'+4m_1p_0^2+4m_1p_0'^2-4m_1p_0p_0'.$$

We left in (7) only the terms proportional to m_2^2 and m_2, keeping in mind the determination of contribution to muonium fine structure in the leading order on parameter m_1/m_2. As will soon become evident, we can't restrict in $F(p,p')$ only by terms $\sim m_2^2$. The quasipotentials of the rest amplitudes in

Fig.1 may be constructed in a similar way. They differ from each other due to momentum dependence in muonic denominators and to the kind of functions f_{i1} (i=1,...,6).

The parts of $F_i(p,p')$, proportional to m_2^2, coincide in all six amplitudes. Let remark, that when substitute $\hat{\epsilon} \to \gamma_5$ in projector operator (6) ($^1S_0$ state), we obtain the same function $f_{12}(p,p')$, as for the $^3S_1$ muonium. This means, that the muonium hyperfine splitting appears as effect of higher order on m_1/m_2. Functions f_{i1} are equal to:

$$f_{21} = -10m_1p'^2 - 5p_0p'^2 + 10m_1pp' + 4p_0pp' - 4p_0'pp' + 2m_1p^2 + 4p_0'p^2 + 12p_0m_1^2- \tag{9}$$
$$-6m_1^2p_0' + 4m_1p_0^2 + 8m_1p_0p_0' - 8m_1p_0'^2 + 4p_0'p_0^2,$$

$$f_{31} = 2m_1^2p'^2 + 4p_0p'^2 + 10m_1pp' - 4p_0pp' + 4p_0'pp' - 10m_1p^2 - 5p_0'p^2- \tag{10}$$
$$-6m_1^2p_0 + 12m_1^2p_0' - 8m_1p_0^2 + 8m_1p_0p_0' + 4m_1p_0'^2 + 4p_0p_0'^2,$$

$$f_{41} = 2m_1p'^2 + p_0p'^2 - 2m_1pp' + 4p_0pp' + 2p_0'pp' - 10m_1p^2 - 8p_0'p^2 - 12m_1^2p_0+ \tag{11}$$
$$+6m_1^2p_0' + 4m_1p_0^2 - 4m_1p_0p_0' + 4m_1p_0'^2 - 8p_0'p_0^2,$$

$$f_{51} = -10m_1p'^2 - 8p_0p'^2 - 2m_1pp' + 2p_0pp' + 4p_0'pp' + 2m_1p^2 + p_0'p^2 + 6m_1^2p_0- \tag{12}$$
$$-12m_1^2p_0' + 4m_1p_0^2 - 4m_1p_0p_0' + 4m_1p_0'^2 - 8p_0'^2p_0,$$

$$f_{61} = -10m_1p'^2 - 5p_0p'^2 - 2m_1pp' - 4p_0pp' - 4p_0'pp' - 10m_1p^2 - 5p_0'p^2- \tag{13}$$
$$-6m_1^2p_0 - 6m_1^2p_0' - 8m_1p_0^2 - 4m_1p_0p_0' - 8m_1p_0'^2.$$

Integral function in (4) has simple poles on loop energies p_0, p_0' in electron, muon and photon propagators. So, the most natural way of integration (4) consists in the calculation of integrals on p_0, p_0' at the first step, using the method of residues. But such an approach of calculation leads, nevertheless, to rather complicated intermediate expressions, what makes highly questionable its subsequent analytical integration on spatial momenta $\vec{p}, \vec{p}'$. So, we have used different approach of integration in (4), connected with transformation of muonic denominators, accounting the necessary calculational accuracy on parameter m_1/m_2. Considering that the spatial momentum of muonic motion in the intermediate state $|\vec{p}| < m_2$, we obtain:

$$D_\mu(p) = p^2 - w^2 + 2m_2p_0 \approx 2m_2\left(p_0 - \frac{\vec{p}^2 + w^2}{2m_2} + i\epsilon\right) \approx 2m_2(p_0 + i\epsilon), \tag{14}$$

where the second approximate equality means that we neglect by the muon kinetic energy in the intermediate state. Doing so, we suppose that the integration contour on variable p_0 must be closed in the lower halfplane. Considering the terms, proportional to m_2^2 in the numerators of all six diagrams (function $f_{12}(p,p')$), we have arrived to the need of expression transformation, which includes the sum of muonic denominators (5). Using the second approximate equality from (14), we obtain:

$$\frac{1}{D_\mu(-p)D_\mu(-p')} + \frac{1}{D_\mu(-p)D_\mu(p'-p)} + \frac{1}{D_\mu(-p')D_\mu(p-p')}+$$
$$+\frac{1}{D_\mu(p)D_\mu(p-p')} + \frac{1}{D_\mu(p')D_\mu(p'-p)} + \frac{1}{D_\mu(p')D_\mu(p)} \approx$$
$$\approx \frac{(-2\pi i)\delta(p_0)}{2m_2}\frac{(-2\pi i)\delta(p_0')}{2m_2}. \tag{15}$$

In the energy spectrum the expression (15) will cause the corrections of order $O(\alpha^4)$, which are canceled by the similar terms from the iteration part of the quasipotential. Consequently, to find the necessary contribution of order α^6, we must use first approximate equality in (14). Taking the difference

$$\frac{1}{2m_2\left(p_0 - \frac{\vec{p}^2+w^2}{2m_2} + i\epsilon\right)} - \frac{1}{2m_2(p_0+i\epsilon)} \approx \frac{(\vec{p}^2 + w^2)}{4m_2^2(p_0+i\epsilon)^2}, \tag{16}$$

let represent the quantity $1/D_\mu(p)$ in the form:

$$\frac{1}{D_\mu(p)} \approx \frac{1}{2m_2(p_0+i\epsilon)} + \frac{(\vec{p}^2+w^2)}{4m_2^2(p_0+i\epsilon)^2}. \tag{17}$$

The second addendum on the r.h.s of (17) is of higher order on m_1/m_2 in comparison with the first one. But it leads to the necessary order correction on the other parameter α. Using the splitting (17), in the sum (15), let extract the terms, which generate the correction $O(\alpha^6)$ and $O(\alpha^6\ln\alpha)$ in the energy spectrum. We may write them in the following manner:

$$\frac{(\vec{p}'^2+w^2)}{8m_2^3}\left[\frac{2\pi i\delta(p_0')}{(p_0+i\epsilon)^2} - \frac{2\pi i\delta(p_0'-p_0)}{(p_0+i\epsilon)^2} - \frac{2\pi i\delta(p_0)}{(p_0'+i\epsilon)^2}\right] + \tag{18}$$
$$+\frac{(\vec{p}^2+w^2)}{8m_2^3}\left[\frac{-2\pi i\delta(p_0')}{(p_0+i\epsilon)^2} - \frac{2\pi i\delta(p_0'-p_0)}{(p_0+i\epsilon)^2} + \frac{2\pi i\delta(p_0)}{(p_0'+i\epsilon)^2}\right] +$$
$$+\frac{(\vec{p}-\vec{p}')^2+w^2}{8m_2^3}\left[-\frac{2\pi i\delta(p_0')}{(p_0+i\epsilon)^2} + \frac{2\pi i\delta(p_0'-p_0)}{(p_0+i\epsilon)^2} - \frac{2\pi i\delta(p_0)}{(p_0'+i\epsilon)^2}\right]$$

It is evident from the three-photon interaction amplitude of the type (6), that the parts of (18) give the necessary order corrections on α in the studied fine structure intervals. The same order corrections $O(m_1/m_2)$, as well as (18), will arise from the quasipotential terms containing the functions $f_{i1}(p,p')$, when we use the second approximation of (14) for muonic denominators. To do definite conclusion about the order of appearing terms in energy spectrum, which are determined by these quasipotential addenda, let transform them for greater simplification. Let consider for definiteness massless terms in the function $f_{i1}(p,p')$, proportional to $\sim p^2, p'^2, pp'$:

$$3p^2\left[\frac{1}{D_\mu(p)} + \frac{1}{D_\mu(-p)} - \frac{1}{D_\mu(p-p')} - \frac{1}{D_\mu(p'-p)}\right] + \tag{19}$$
$$+3p'^2\left[\frac{1}{D_\mu(-p')} + \frac{1}{D_\mu(p')} - \frac{1}{D_\mu(p'-p)} - \frac{1}{D_\mu(p-p')}\right] -$$
$$-6pp'\left[\frac{1}{D_\mu(-p')} + \frac{1}{D_\mu(p')} + \frac{1}{D_\mu(-p)} + \frac{1}{D_\mu(p)} - \frac{1}{D_\mu(p'-p)} - \frac{1}{D_\mu(p-p')}\right] \approx$$
$$\approx \frac{3p^2}{2m_2}\left[-2\pi i\delta(p_0) + 2\pi i\delta(p_0-p_0')\right] + \frac{3p'^2}{2m_2}\left[-2\pi i\delta(p_0') + 2\pi i\delta(p_0-p_0')\right] -$$
$$-\frac{6pp'}{2m_2}\left[-2\pi i\delta(p_0) - 2\pi i\delta(p_0') + 2\pi i\delta(p_0-p_0')\right].$$

The transformation of other terms in functions $f_{i1}(p,p')$ may be carried out in the same manner. The next step of calculation consists in an integration over four-momenta in expressions (18)-(19). The typical two-loop integral, that results on this way has the following structure [16]:

$$K_i = (4\pi)^2\int\frac{d^4pd^4p'}{-(2\pi)^8}\frac{G_i(p_0',p_0,m_1)P(\vec{p},\vec{p}',w)}{(p'^2-w^2+i\epsilon)[(p-p')^2+i\epsilon](p^2-w^2+i\epsilon)D_e(p')D_e(p)}, \tag{20}$$

where $G_i(p_0',p_0,m_1)$ contains one δ - function, and $P(\vec{p}',\vec{p},w)$ is a polynom. To calculate fundamental integrals (20) we have used the Feynman parameterization in order to combine the denominators of the particle propagators, and the symmetry properties of the integral with the replacement $p \Leftrightarrow p'$. There are the next set of functions $G_i(p_0',p_0,m_1)$, which appear in this paper:

$$G_1 = -\frac{2\pi i\delta(p_0-p_0')2m_1}{(p_0+i\epsilon)^2}, \quad G_2 = -\frac{2\pi i\delta(p_0)2m_1}{(p_0'+i\epsilon)^2}, \quad G_3 = -2\pi i\delta(p_0)2m_1, \tag{21}$$
$$G_4 = -2\pi i\delta(p_0')2m_1, \quad G_5 = -2\pi i\delta(p_0-p_0')2m_1.$$

The results of integrations for K_i (20) are presented in Table (see also [16]).

Table of the integrations K_i (20), appearing in the muonium fine structure calculations

	$\vec{p}^2(\vec{p}\vec{p}')$	$(\vec{p}\vec{p}')^2$	$\vec{p}'^2(\vec{p}\vec{p}')$	$w^2(\vec{p}\vec{p}')$
K_1	$2\ln 2-\frac{1}{2}$	$2\ln 2-\frac{1}{2}$	$2\ln 2-\frac{1}{2}$	0
	$\vec{p}^2(\vec{p}'^2-\vec{p}\vec{p}')$	$\vec{p}\vec{p}'(\vec{p}'^2-\vec{p}\vec{p}')$	$w^2(\vec{p}'^2-\vec{p}\vec{p}')$	$w^2\vec{p}^2$
K_2	$\frac{1}{2}\ln\frac{m_1}{2w}-\frac{1}{32}$	$\frac{1}{4}\ln\frac{m_1}{2w}-\frac{13}{32}$	$\frac{5}{32}$	$\frac{2}{3}$
	$\vec{p}'^2$	$\vec{p}\vec{p}'$	$\vec{p}^2$	w^2
K_3	—	$\frac{1}{4}\ln\frac{m_1}{2w}-\frac{1}{4}$	$\frac{1}{2}\ln\frac{m_1}{2w}-\frac{1}{8}$	$\frac{1}{8}$
K_4	$\frac{1}{2}\ln\frac{m_1}{2w}-\frac{1}{8}$	$\frac{1}{4}\ln\frac{m_1}{2w}-\frac{1}{4}$	—	$\frac{1}{8}$
K_5	$\ln 2$	$\ln 2-\frac{1}{2}$	$\ln 2$	0

Arter collecting the contributions, defined by expressions (8-13) and (18) we obtain:

$$\Delta B_1=-\frac{1}{2}(Z\alpha)^6\frac{m_1^2}{m_2} \tag{22}$$

$$\Delta B_2=(Z\alpha)^6\frac{m_1^2}{m_2}(6\ln 2-\frac{11}{48}) \tag{23}$$

We have introduced in (20) the photon mass w to avoid "infrared" singularities. The "infrared" logarithms $\ln w$, containing this photon mass (see table of integrals K_i), and appearing at intermediate expressions, are mutually cancelled in the corrections ΔB_1, ΔB_2.

Let consider now the quasipotential addenda, containing the momenta of particle relative motion in the initial and final states. We denote them by $\vec{r}_1$ and $\vec{r}_2$ correspondingly. Their consideration leads to modification of f_{i1}, which acquire the following additional terms:

$$\Delta f_{21}=10m_1p'r_2+5p_0p'r_2+m_1pr_2+3p_0'pr_2, \tag{24}$$

$$\Delta f_{31}=-m_1p'r_1-3p_0p'r_1-10m_1pr_1-5p_0'pr_1, \tag{25}$$

$$\Delta f_{41}=m_1(7p'r_1+5p'r_2+10pr_1+5pr_2)+6p_0p'r_1+5p_0p'r_2+8p_0'pr_1+5p_0'pr_2, \tag{26}$$

$$\Delta f_{51}=m_1(-5p'r_1-10p'r_2-5pr_1-7pr_2)-5p_0p'r_1-8p_0p'r_2-5p_0'pr_1-6p_0'pr_2, \tag{27}$$

$$\Delta f_{61}=m_1(-11p'r_1-11p'r_2-11pr_1-11pr_2)-8p_0p'r_1-8p_0p'r_2-8p_0'pr_1-8p_0'pr_2. \tag{28}$$

Using again the symmetry properties of appearing integrals under simultaneous variable replacement $p\Leftrightarrow p'$, $r_1\Leftrightarrow r_2$, we obtain cancellation of all integrations in (24)-(28). So, the contribution the particle relative motion in the fine structure with the accuracy $O(m_1/m_2)$ is equal to zero. Thus the full value of the calculated correction $(Z\alpha)^6m_1^2/m_2$ for hydrogen-like system S-states is defined as a sum of expressions (22) and (23):

$$\Delta B=(Z\alpha)^6\frac{1}{n^3}\frac{\mu^3}{m_1m_2}\left(6\ln 2-\frac{35}{48}\right). \tag{29}$$

The numerical value of (29) for the muonium fine structure interval $2^3S_1\div 1^3S_1$ takes the value 0,271 =8CHz.

To conclude, we applyed in this paper the diagrammatic approach to calculate the corrections of order $(Z\alpha)^6m_1^2/m_2$. We performed the most complicated part of calculations, connected with the two-loop integrations. In order to obtain the total value of the necessary order contribution it is important to complete these results by similar corrections from one-photon, two-photon interactions as well as by the results from the second order perturbation theory [18]. It is the aim of future studies.

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