S-matrix poles and unusual resonance phenomena

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Abstract

A sharp peak in the cross section is usually interpreted as a resonance, especially if it reaches the unitarity limit and can be reasonably well fitted by a Breit-Wigner shape. We shall use a simple example to show that this interpretation is not always correct. An apparent peak may not be a true resonance, and an isolated S-matrix pole does not necessarily generate a Breit-Wigner shape.

We begin this note by presenting a histogram of computer generated fake "experimental data." We shall shortly explain how this "data" was created. The histogram in Fig. 1(a) shows a prominent peak near $E = 9.9$ units. To analyze this further, it is natural to subtract a "background." The result is given in Fig. 1(b). The resulting peak looks very like a strong resonance, and therefore we try to fit it with a Breit-Wigner shape. Figure 1(c) shows the results of such an attempt (we adjusted the parameters to obtain a reasonable fit without trying to find the best possible fit).

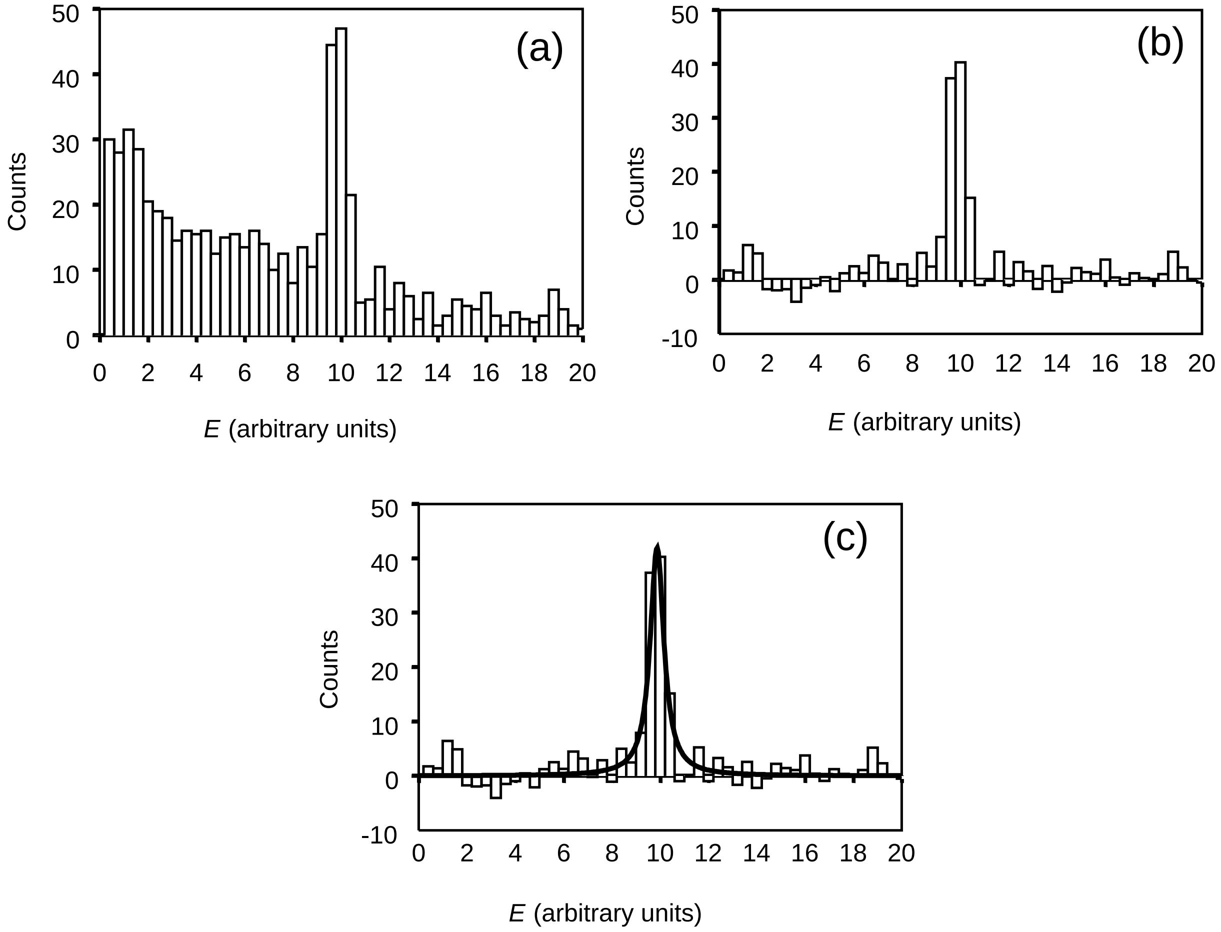

Figure 1: (a) Computer generated fake experimental data. (b) Fake data after removal of a smooth background. (c) Breit-Wigner fit to the data of Fig. 1(b) (no attempt was made to optimize the fit).

It is now time to explain how Fig. 1 was generated. We began with the potential shown in Fig. 2. This is a truncated "von Neumann-Wigner" potential [1]. If the potential had not been cut off, the potential would have supported an s-wave bound state at energy $E = 9.87$ units, embedded in the continuous spectrum. It is possible to solve the s-wave scattering theory exactly for the truncated potential. We added a smoothly varying background to the s-wave scattering cross section, and then used a random number generator and the binomial distribution, with relative probabilities proportional to our artificially created "cross section," to create the histogram of Fig. 1(a). When the background is removed again as in Fig. 1(b), we obtained a histogram which should approximate the exactly calculated s-wave cross section.

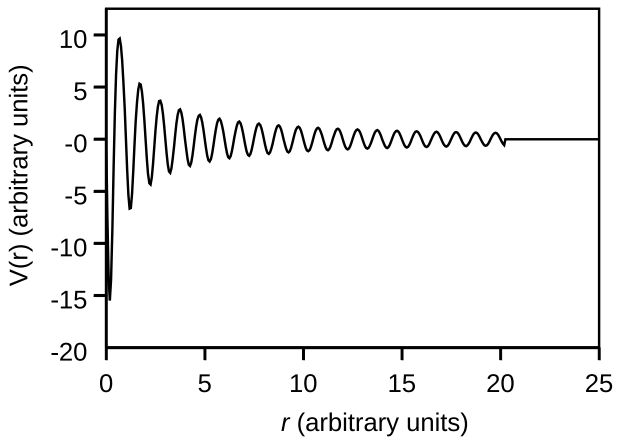

Figure 2: The truncated von Neumann-Wigner potential used to create the fake data.

Nobody would be surprised if truncating the von Neumann-Wigner potential converted the bound state embedded in the continuum into a conventional resonance, as suggested by Fig. 1(c). Indeed, the exact S-matrix has a complex pole at $E_0 - \frac{1}{2}i\Gamma = 9.8694 - 0.009\,758\,i$ units, and has no other nearby singularities, as would be expected for a resonance.

However this is not really what happens. Figure 3(a) shows the (suitably scaled) exact cross section, and Fig. 3(b) shows it superimposed on the histogram of Fig. 1(b). Instead of a single resonance peak, the cross section has twin peaks separated by a very narrow gap, and is strictly zero at $E_1 = 9.86953$ units. Furthermore, the resonance width predicted by Γ turns out to be the width of the gap, not the width of the whole twin peak structure. What is going on?

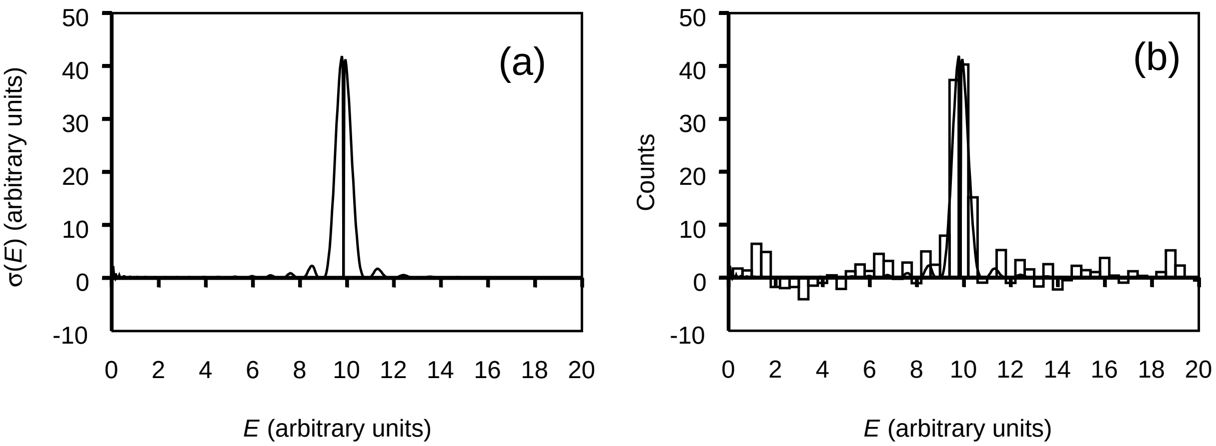

Figure 3: (a) s-wave cross section calculated exactly from the potential of Fig. 2. (b) The computed cross section superimposed on the data of Fig. 1(b).

We explain the phenomenon as follows. The s-wave partial cross section is

$$\sigma(E) = \frac{4\pi}{E}\sin^2\delta(E) = \frac{\pi}{E}\left|S(E) - 1\right|^2 . \tag{1}$$

The S-matrix is unitary. Hence if there is a pole at $E = E_0 - \frac{1}{2}i\Gamma$ we may write

$$S(E) = \frac{E - E_0 - \frac{1}{2}i\Gamma}{E - E_0 + \frac{1}{2}i\Gamma} \cdot \bar{S}(E), \tag{2}$$

where $\bar{S}(E)$ is also unitary. If $\bar{S}(E) \approx 1$ near $E = E_0$ then

$$\sigma(E) \approx \frac{4\pi}{E_0} \cdot \frac{\frac{1}{4}\Gamma^2}{(E - E_0)^2 + \frac{1}{4}\Gamma^2}, \tag{3}$$

the familiar Breit-Wigner formula for the cross section close to a resonance.

If, however, $\sigma(E_1) = 0$ then

$$\bar{S}(E_1) = \frac{E_1 - E_0 + \frac{1}{2}i\Gamma}{E_1 - E_0 - \frac{1}{2}i\Gamma}. \tag{4}$$

If also $E_1 \approx E_0$ and if $\bar{S}(E) \approx$ constant for $E \approx E_1 \approx E_0$ (as is to be expected if the pole at $E_0 - \frac{1}{2}i\Gamma$ is isolated) then Eqs. (1), (2) and (4) yield

$$\sigma(E) \approx \frac{(E - E_1)^2}{(E_1 - E_0)^2 + \frac{1}{4}\Gamma^2} \times \frac{4\pi}{E_0} \cdot \frac{\frac{1}{4}\Gamma^2}{(E - E_0)^2 + \frac{1}{4}\Gamma^2} \approx \frac{4\pi}{E_0} \cdot \left[1 - \frac{\frac{1}{4}\Gamma^2}{(E - E_0)^2 + \frac{1}{4}\Gamma^2}\right], \tag{5}$$

so that the Breit-Wigner shape given by Eq. (3) is multiplied by an additional factor which vanishes when $E = E_1$. The second form of Eq. (5) is obtained from the first form by setting $E_1 = E_0$, and shows that the cross section predicted by this approximation is very close to an *inverted* Breit-Wigner in shape. We refer to the first form of Eq. (5) as the *pole approximation.* We still wish to call this a "resonance phenomenon," even though the structure does not have a conventional Breit-Wigner shape, because like conventional resonances the phenomenon is associated with a well isolated pole of the S-matrix.

The relevance of this discussion for the problem at hand is illustrated by Fig. 4. This shows the cross section for the energy region very close to the pole, computed exactly using the potential of Fig. 2 and computed using the pole approximation with the values of E_0, Γ, and E_1 given earlier. As already noted, if $E_1 \approx E_0$ then $\sigma(E)$ has a shape almost exactly like an inverted Breit-Wigner shape. This can be seen most dramatically by turning the page around and looking at Fig. 4 upside down.

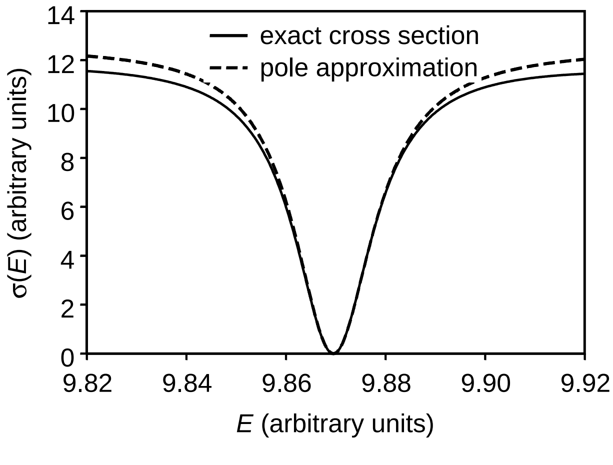

Figure 4: The predictions of the "pole approximation" and the exact cross section for the gap between the twin peaks.

What is the cause of this phenomenon? One might ask whether it is a result of the many oscillations in the potential illustrated in Fig. 2. In fact, this is not so. If the von Neumann-Wigner potential is cut off at a smaller radius, the overall width of the twin peak structure in the cross section increases, but the S-matrix still has a zero close to the position of the isolated S-matrix pole. Therefore once

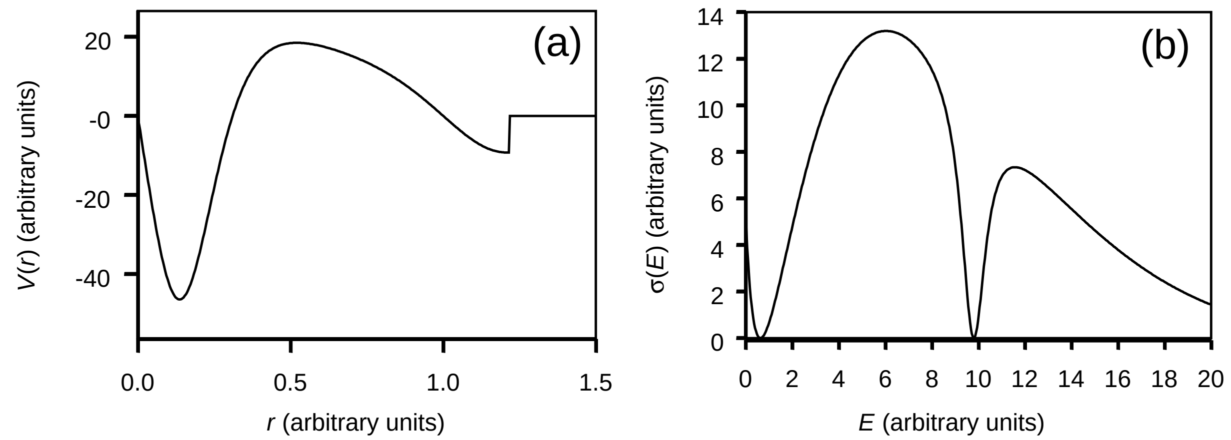

Figure 5: (a) A severely truncated von Neumann-Wigner potential. (b) The s-wave scattering cross section computed exactly from the potential of (a).

again the cross section has a shape similar to that of an inverted Breit-Wigner. This is illustrated, for an extreme case, by Fig. 5. Because the von Neumann-Wigner potential is so severely truncated, the inverted Breit-wigner shape is somewhat distorted but is still clearly present.

We deliberately chose a large cutoff for the potential of Fig. 2 in order to make the twin peak structure very narrow. As a result, the gap between the peaks was impossible to resolve after the theoretical cross section had been converted into a histogram (with the addition of some fake statistical error). The lessons to be learned from all this are first that an experimentally observed peak in a cross section is not necessarily a conventional Breit-Wigner resonance, and second that if two peaks very close together are observed in a cross section the gap between them may well be more significant than the peaks themselves. An apparent single peak may instead really be unresolved twin peaks, and the gap between twin peaks may indicate the presence of a *single* well isolated pole of the S-matrix, whose contribution to the S-matrix is interfering destructively with a relatively constant "background" $\bar{S}\,(E)$ which happens to be almost pure imaginary.

A multichannel version of our analysis is given by Weinberg [2]. Weinberg mentions the possibility of the "inverted Breit-Wigner" phenomenon, but offers no examples. As noted by Weinberg, the "inverted Breit-Wigner" phenomenon is closely related to the Ramsauer-Townsend effect [3] in electron scattering by atoms.

A fuller discussion of the s-wave scattering due to a truncated von Neumann-Wigner potential has been submitted to *Physical Review* A [4].

References

[1] For recent treatments of von Neumann-Wigner potentials, see T. A. Weber and D. L. Pursey, Phys. Rev. A **50**, 4478 (1994) and references cited therein.

[2] Steven Weinberg, *The Quantum Theory of Fields*, vol. 1, pp. 159–165 (Cambridge University Press, 1995).

[3] See R. Kollath, Phys. Z. **31**, 985 (1931).

[4] T. A. Weber and D. L. Pursey, "Scattering from a truncated von Neumann-Wigner potential," submitted to Phys. Rev. A (1997).