$\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence at finite temperature

L. Chekhov [1]

Steklov Mathematical Institute
Gubkin st. 8, 117966, GSP-1, Moscow, Russia

Abstract

The AdS/CFT correspondence is established on the correlation function level for the AdS_3 space compactified on a solid torus with the CFT field on the boundary. The BTZ black hole solutions in AdS_3 are T-dual to the solution in the AdS_3 space without singularity.

1 Introduction

The AdS/CFT correspondence [1, 2, 3] has been verified for interacting fields [4, 5] and also in cases where the space–time geometry is more involved than the spherical one using various approaches (see [6, 7, 8].

In [8], the massless scalar field theory on AdS_3 space was considered. (The method can be generalized to the massive modes of $\mathrm{AdS}_n \times S^{d+1}$ space.) Here we consider the case of a torus and discuss the physical content of the results obtained. We also consider the case of a homogeneously compactified AdS_3 manifold without (topological) singularity in the interior (as there is no boundary as $s \to \infty$; our calculations technically resemble the bulk calculations of [5]). We show that the classical scalar field theory on the AdS_3 manifold in the bulk gives us the appropriate quantum correlators on the boundary.

The compactification ideologically corresponds to considering the finite temperature case, so we may interpret the results as correlators of boundary fields for the AdS_3 space at finite temperature [9]. Following [9], one must take into account *all possible* solutions of the Einstein gravity that have the anti-de Sitter metric at infinity. In the AdS_3 case, the black-hole solutions of Hawking and Page [10, 9] are *T-dual* [11, 12] to the case of a pure AdS_3 space without internal singularities; this holds for the corresponding correlation functions as well. It turns out that the case where a conical singularity (the angle deficit) is developed in the vicinity of a (unique) closed geodesic inside the solid torus can be treated as well; the correlation functions were found for the case where the angle deficit is a rational number [13].

In Sec. 2, we recall the general structure of AdS_3 manifolds and introduce the ε-cone regularization in order to make the volume and the boundary area finite. In Sec. 3, we review the relationship between the black hole and empty space solutions of the AdS_3 gravity. In Sec. 4, we formulate the scalar field theory in AdS_3, while the corresponding correlation functions are calculated by using an improved technique in Sec. 5. A brief discussion is in Sec. 6.

2 Geometry of AdS_3 manifolds

The group $SL(2,\mathbf{C})$ of conformal transformations of the complex plane admits the continuation to the upper half-space $\mathbf{H}_3^+$ endowed with the constant negative curvature (AdS_3 space). In the Schottky uniformization picture, Riemann surfaces of higher genera can be obtained from $\mathbf{C}$ by factorizing it over a finitely generated free-acting discrete subgroup $\Gamma \subset SL(2,\mathbf{C})$. Therefore, we can continue the action of this subgroup to the whole AdS_3 and, upon factorization, obtain a three-dimensional manifold of constant negative curvature (an AdS_3 manifold) whose boundary is (topologically) a two-dimensional Riemann surface [14, 15].

We consider the simplest case of a genus one AdS_3 manifold obtained after the identification

$$(w,\overline{w},s) \sim (qw,\overline{qw},s|q|), \tag{2.1}$$

where $q = \mathrm{e}^{\,a+ib}$ is the modular parameter, $a,b \in \mathbf{R},\ a>0$, and $w,\overline{w} = x+iy,\ x-iy$ are the coordinates on $\mathbf{C}$, while s is the third coordinate on AdS_3.

[1] email: chekhov@mi.ras.ru

Adopting the AdS/CFT correspondence approach, we must first regularize expressions in order to make them finite. In the AdS_n case with the brane singularity at infinity, this was done by setting the boundary data on the ε-plane [2] rather than at infinity (zero plane). However, in our case, an ε-plane is not invariant w.r.t. transformations (2.1). Instead, we can set the boundary data on the *ε-cone*—the set of points

$$z/r = \varepsilon, \quad r^2 = w\overline{w} + s^2.$$

Given the boundary data on this cone, we fix the problem setting and the Laplace equation then has a unique solution (the Dirichlet problem on a compact manifold).

Geometrically, performing the ε-cone regularization and factorizing over the group Γ of transformations (2.1), we obtain the solid torus on whose boundary (the two-dimensional torus) the CFT fields dwell. The "center" of the torus is a unique closed geodesic, which has the length $\log|q|$ (the image of the vertical half-line $z = \overline{z} = 0$), while the AdS-invariant distance r from this geodesic to the image of the ε-cone is constant, $\cosh r = 1/\varepsilon$.

3 Black hole solution in $\mathbf{AdS_3}$ in spherical coordinates

It is convenient to reformulate the free-field problem on AdS_3 in the spherical coordinates $r \equiv \mathrm{e}^{\tau}, \theta, \varphi$ [11] in which relations (2.1) read

$$\begin{aligned} &(\tau, \theta, \varphi) \sim (\tau + ma, \theta, \varphi + mb + 2\pi n), \qquad m, n \in \mathbf{Z}, \\ &dl^2 = \left(d\tau^2 + d\theta^2 + \sin^2\theta\, d\varphi^2\right)/\cos^2\theta, \end{aligned} \tag{3.1}$$

where dl^2 is the invariant interval squared. On the cone $\sin\theta = \varepsilon$, the usual toric periodic conditions are imposed on fields depending on the 2D variables τ and φ. Upon the identification

$$\rho = b\tan\theta, \quad t = b\log r, \quad \varphi = \varphi \tag{3.2}$$

we obtain the AdS_3 metric in the standard form [10],

$$ds^2 = \left(1 + \frac{\rho^2}{b^2}\right) dt^2 + \frac{1}{1 + \rho^2/b^2} d\rho^2 + \rho^2 d\varphi^2, \tag{3.3}$$

with fields to be regular at $\rho = 0$.

The remarkable fact is that the *spinning* black hole solution metric (with the nonzero angular momentum J and mass M) given by the interval

$$\begin{aligned} &ds^2 = (N^{\perp})^2 d\rho^2 + (N^{\perp})^{-2} dr^2 + r^2(d\phi + N^{\phi} d\rho)^2, \\ &N^{\perp} = \left(-M + \frac{r^2}{l^2} - \frac{J^2}{4r^2}\right)^{1/2}, \qquad N^{\phi} = -\frac{J}{2r^2}, \end{aligned} \tag{3.4}$$

with the following indentifications imposed:

$$\begin{aligned} &\phi \sim \phi + 2\pi, \qquad \begin{pmatrix} \rho \\ \phi \end{pmatrix} \sim \begin{pmatrix} \rho \\ \phi \end{pmatrix} + \frac{2\pi}{M} \begin{pmatrix} r_+ \\ |r_-|/l \end{pmatrix}, \\ &r_\pm^2 = \frac{Ml^2}{2}\left[1 \pm \left(1 + \frac{J^2}{M^2 l^2}\right)^{1/2}\right], \end{aligned} \tag{3.5}$$

can be reduced by the coordinate transformations

$$\varphi = \frac{r_+}{l^2}\rho + \frac{|r_-|}{l}\phi, \quad \tau = \frac{r_+}{l}\phi - \frac{|r_-|}{l^2}\rho, \quad \theta = \arcsin\sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} \tag{3.6}$$

to coordinates (3.1) with $a = 2\pi r_+/l$ and $b = 2\pi|r_-|/l$ [11].

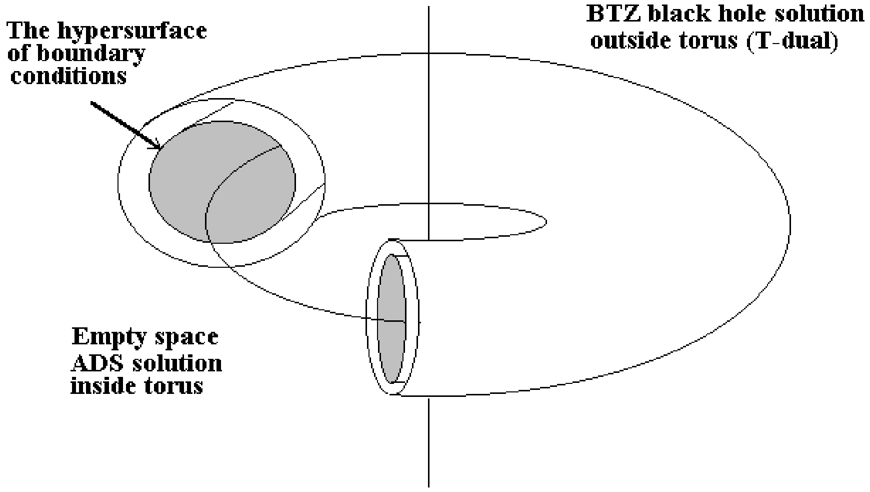

Figure 1:

Therefore, metric (3.4) with the respective (two-dimensional) radial and angular coordinates ρ and ϕ becomes metric (3.1) with the respective coordinates τ and φ. Note that (for $J = 0$) radial and angular coordinates readily interchange and we obtain the torus that is T-dual to the initial one (see Fig. 1 where our problem setting is depicted readily).

4 Scalar field on AdS$_3$ in spherical coordinates

The action of the massless scalar field Φ on AdS$_3$ in coordinates (3.1) is

$$\int \frac{dx\,dy\,ds}{s^3}\left\{s^2\partial_s\Phi\partial_s\Phi + s^2(\partial_x\Phi\partial_x\Phi + \partial_y\Phi\partial_y\Phi)\right\} =$$

$$= \int \tan\theta d\tau\, d\theta\, d\varphi \left\{\partial_\tau\Phi\partial_\tau\Phi + \partial_\theta\Phi\partial_\theta\Phi + \frac{1}{\sin^2\theta}\partial_\varphi\Phi\partial_\varphi\Phi\right\} \tag{4.1}$$

It admits the variable separation,

$$\Phi(\tau,\theta,\varphi) = \sum_{k,m\in\mathbf{Z}} \Phi_{k,m}(\tau)Y_{k,m}(\sin\theta)X_m(\varphi). \tag{4.2}$$

Taking into account the periodicity conditions

$$\Phi(\tau + \log|q|, \theta, \varphi + \arg q) = \Phi(\tau,\theta,\varphi), \qquad \Phi(\tau,\theta,\varphi+2\pi) = \Phi(\tau,\theta,\varphi), \tag{4.3}$$

where q is the modular parameter of the torus,

$$q = \mathrm{e}^{\,a+ib}, \quad a, b \in \mathbf{R}, \tag{4.4}$$

we obtain

$$X_m(\varphi) = \mathrm{e}^{\,im\varphi}, \qquad \partial_\varphi^2 X_m = -m^2 X_m; \tag{4.5}$$

$$\Phi_{k,m}(\tau) = \mathrm{e}^{\,i\left[-\frac{mb}{a}+\frac{2\pi k}{a}\right]\tau}, \qquad \partial_\tau^2\Phi_{k,m}(\tau) = -\frac{w^2}{4}\Phi_{k,m}(\tau), \tag{4.6}$$

$$w \equiv -\frac{mb}{2a} + \frac{\pi k}{a}. \tag{4.7}$$

For the function $Y_{k,m}(\sin\theta)$, the eigenvalue problem is

$$\frac{\cos\theta}{\sin\theta}\partial_\theta\left(\frac{\sin\theta}{\cos\theta}\partial_\theta Y_{k,m}(\sin\theta)\right)-\frac{m^2}{\sin^2\theta}Y_{k,m}(\sin\theta)=\frac{w^2}{4}Y_{k,m}(\sin\theta). \tag{4.8}$$

Considering the problem on the interval $1-\varepsilon\geq\sin(\theta)\geq 0$ with the regularity condition at $\theta=0$, we can reduce Eq. (4.8) to the standard hypergeometric equation whose general solution that is regular at $\theta=0$ yields

$$\begin{aligned}\Phi(\tau,\theta,\varphi) &= \sum_{m,k\in\mathbf{Z}} \mathrm{e}^{\,im\varphi}\,\mathrm{e}^{\,i\left[-\frac{mb}{a}+\frac{2\pi k}{a}\right]\tau}\times C_{k,m}\times Y_{k,m}(\sin\theta),\\ Y_{k,m}(\sin\theta) &= [\sin\theta]^{|m|}\,{}_2F_1\left(\frac{|m|}{2}+i\frac{-mb+2\pi k}{2a},\frac{|m|}{2}-i\frac{-mb+2\pi k}{2a};|m|+1;\sin^2\theta\right),\end{aligned} \tag{4.9}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric series,

$${}_2F_1(a,b;c;z)\equiv\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\cdot k!}z^k,\qquad (a)_k\equiv\prod_{i=0}^{k-1}(a+i),$$

and $C_{k,m}$ are the mode amplitudes.

Expression (4.9) is singular at $z=1$ and we must find its asymptotic behavior for $z=1-\varepsilon$. The analysis produces action (4.1) in the form of the mode expansion ($C^*_{k,m}\equiv C_{-k,-m}$) [8]

$$\begin{aligned}-\sum_{k,m\in\mathbf{Z}}|C_{k,m}|^2\left(\frac{m^2}{4}+w^2\right)\big[\log\varepsilon+\Psi\left(1+m/2+iw\right)+\Psi\left(1+m/2-iw\right)+\\ +\Psi\left(1-m/2+iw\right)+\Psi\left(1-m/2-iw\right)-4\Psi(1)\big],\end{aligned} \tag{4.10}$$

where w and q are in (4.7), (4.4).

5 Correlation function

In expression (4.10), the mode multiplier is the Fourier transform of the correlation function of the boundary CFT field on the torus. We are interested in the nonlocal contribution to Green's functions on torus coming from this formula and disregard all local terms, which can be removed by a proper renormalizing procedure.

The m-dependence in (4.10) is analytic [8] and we may represent the Ψ-function using the formula

$$\Psi(1+a)-\Psi(1+b)=\sum_{j=0}^{\infty}\left(\frac{1}{1+j+b}-\frac{1}{1+j+a}\right).$$

We now find the CFT correlation functions that follow from (4.10). The term $\left(\frac{m^2}{4}+w^2\right)$ in front of the summand is the Laplacian action, and the correlation function is

$$\begin{aligned}G(\tau,\varphi) &= \frac{1}{4}\big(\partial^2_\varphi+\partial^2_\tau\big)\sum_{m,k\in\mathbf{Z}}\mathrm{e}^{\,im\varphi+i\left[-\frac{mb}{a}+\frac{2\pi k}{a}\right]\tau}\left[2\log\varepsilon+\sum_{(\pm),\pm}\Psi\big(1(\pm)m/2\pm iw\big)-4\Psi(1)\right]=\\ &= \frac{1}{4}\big(\partial^2_\varphi+\partial^2_\tau\big)\sum_{m,k\in\mathbf{Z}}\mathrm{e}^{\,im\varphi+i\left[-\frac{mb}{a}+\frac{2\pi k}{a}\right]\tau}\times\\ &\quad\times\left[2\log\varepsilon+\sum_{(\pm),\pm}\sum_{l=1}^{\infty}\left(\frac{4}{l}-\frac{1}{l(\pm)\frac{m}{2}\pm i\big(-\frac{mb}{2a}+\frac{\pi k}{a}\big)}\right)\right].\end{aligned} \tag{5.1}$$

The sign $\sum_{(\pm),\pm}$ in (5.1) denotes the sum over four terms with all possible appearances of "+" and "−" signs. We distinguish between two appearances of $\pm$ signs by taking one of them in parentheses. Constant and polynomial terms are irrelevant to our discussion as they produce only local contributions.

First, we take the sum over k,

$$\sum_{k=-\infty}^{\infty} \frac{\mathrm{e}^{2\pi i k\tau/a}}{l(\pm)\frac{m}{2} \mp \left(\frac{imb}{2a} - \frac{i\pi k}{a}\right)} = f(\tau/a). \tag{5.2}$$

The function $f(\tau/a)$ is periodic under the shift $\tau \to \tau + 1$ and satisfy the functional equation

$$\pm\frac{1}{2a}\partial_{\tau/a} f(\tau/a) + \left(l(\pm)\frac{m}{2} \mp \frac{imb}{2a}\right) f(\tau/a) = \delta^1_{\Pi}(\tau/a), \tag{5.3}$$

where $\delta^1_{\Pi}(x)$ is the periodic δ-function with the unit period. A (unique) solution to (5.3) that is periodic in τ is a saw-tooth-like exponential curve

$$f(\tau/a) = A\,\mathrm{e}^{\chi\tau/a} \quad \text{for} \quad \tau/a \in (0,1),$$

which is to be continued periodically to the whole $\mathbf{R}$. Equation (5.3) gives

$$\chi = \mp 2a\big(l(\pm)m/2\big) + imb \quad \text{and} \quad A = \frac{\pm 2a}{1 - \mathrm{e}^{\mp 2a\left(l(\pm)m/2\right) + imb}}.$$

Therefore, the remaining nonlocal terms are combined into the sum

$$-\sum_{\substack{m\in\mathbf{Z}\\ l=1}}^{\infty} \mathrm{e}^{im\varphi - imb\tau/a} \sum_{\pm,(\pm)} \frac{\pm 2a\,\mathrm{e}^{\mp 2\left(l(\pm)m/2\right)\tau + imb\tau/a}}{1 - \mathrm{e}^{\mp 2a\left(l(\pm)m/2\right) + imb}}, \quad 0 \le \tau < a. \tag{5.4}$$

Expression (5.4) becomes

$$\sum_{\substack{m\in\mathbf{Z}\\ l=1}}^{\infty} \Re\left[-2a\frac{\mathrm{e}^{mz+2il\varphi}}{1 - q^m\,\mathrm{e}^{2ilb}} + 2a\frac{\mathrm{e}^{mz-2il\varphi}}{1 - q^m\,\mathrm{e}^{-2ilb}}\right], \qquad z \equiv \tau + i\varphi. \tag{5.5}$$

It is convenient to represent the denominators in (5.5) as

$$\frac{1}{1 - q^m\,\mathrm{e}^{i\chi}} = \begin{cases} \sum_{p=0}^{\infty} q^{m\cdot p}\,\mathrm{e}^{ip\chi} & \text{for} \quad |q| > 1,\ m < 0, \\ -\sum_{p=1}^{\infty} q^{-m\cdot p}\,\mathrm{e}^{-ip\chi} & \text{for} \quad |q| > 1,\ m > 0. \end{cases} \tag{5.6}$$

Now, it is easy to sum over l and m in (5.5) (the term with $m = 0$ vanishes). We obtain

$$G(z,\overline{z}) = \frac{1}{4}\partial_z\partial_{\overline{z}}\Im\left\{2a\sum_{p=1}^{\infty}\frac{1}{q^p\,\mathrm{e}^{-z} - 1}\cot(\varphi - bp) - 2a\sum_{p=0}^{\infty}\frac{1}{q^p\,\mathrm{e}^{z} - 1}\cot(\varphi - bp)\right\} \tag{5.7}$$

and it is straightforward to take the derivatives. The answer is

$$G(z,\overline{z}) = \frac{a}{4}\sum_{p=-\infty}^{\infty}\frac{1}{\sinh^2\frac{z+ap+ibp}{2}\sinh^2\frac{\overline{z}+ap-ibp}{2}}. \tag{5.8}$$

This is the correlation function of two Yang–Mills tensor field insertions on torus. At singularity points, it has the proper behavior $G(r) \sim 1/r^4$, which is intrinsic for the weight two CFT fields.

6 Discussion

Having an AdS asymptotic metric in the boundary domain, we can continue it in a bulk in (at least two) different ways: as an empty space or as a space with the BTZ black hole singularity. From the CFT standpoint, these two continuations are connected by the T-duality transformation; therefore, adopting a viewpoint that we must take a sum over all possible continuations of the AdS metric (related, probably, to instantonic modes) for obtaining a proper correlation function on the boundary, we obtain, instead of (5.8), the *sum* of (5.8) and its T-dual.

Obtained expression (5.8) describes the two-point correlator $\langle G^2_{\mu\nu}(0,0)G^2_{\mu\nu}(z,\overline{z})\rangle$ of the Yang–Mills tensor insertions at nonzero temperature, which, from the conformal field theory standpoint, corresponds to the case of a torus. This demonstrates again that the AdS/CFT correspondence holds in our case where no singularity at the AdS time infinity is assumed.

Already in the free field theory, there remain questions on the mass spectrum, on the generalization to higher dimensions, etc. Of special interest is the question how to consider solid Riemann surfaces of higher genera. The construction works well in this case, but the ε-regularized surface, or the boundary of the integation domain, cannot be described in the invariant distance terms (the structure of the set of closed geodesics becomes very involved already starting from genus two); however, we hope that one can obtain a proper answer using an approximation technique. In the present calculations, we disregard all local contributions. However, these contributions becomes important when considering additional boundary terms in the initial action. It would be interesting to check whether the Hamiltonian prescription of [16] holds in this case.

It is also interesting to relate the obtained correlation functions (5.8) with the S-matrix scattering problem in AdS_3 [17].

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