Gravitational radiation from relativistic membrane

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Abstract

We calculate gravitational radiation from an oscillating relativistic membrane in the four-dimensional space-time. A particular excitation mode is chosen to allow the computation to be done analytically. The membrane is supposed to have a toroidal shape with the oscillating smaller radius and approximately constant second one. Angular distribution of radiation is affected by the interference factor caused by the coherence of radiation from different segments of the membrane. It is found that the membrane radiative timelife depends only on the tension parameter and is independent on the membrane size contrary to the case of the radiating string.

1 Introduction and summary

It is well understood now that topological defects which could exist during the very early stage of the cosmological expanstion must leave footprints in the spectrum of background gravitational radiation. Gravitational radiation of cosmic strings was extensively studied during two past decades [1, 2, 3], however their mere existence is now questioned in the light of new measurements. Other defects which could produce considerable amount of gravitational radiation are two-dimentional ones, such as domain walls. Radiation from domain walls was discussed in [4, 5] using some particular field theoretical models for their description. Meorover, one can hope to obtain sensible results already in the framework of purely geometrical approach considering the model of a structureless relativistic membrane (see e.g. [6, 7, 8, 9]). This is the subject of the present work .

We calculate gravitational radiation for a particular shape of an oscillating membrane, however, some properties of the radiation seem to be general enuogh. The feature which is related to the extended nature of the radiating source is the possibility of interference effects due to the coherence of radiation emitted by different parts of the object. We consider a closed membrane of a toroidal shape, such that one radius of the torus is much smaller than the other. Smaller radius is oscillating, while the larger one can be considered approximately constant. We show that these assumptions are consistent with the equations of motion of the membrane which can be solved in the gauge ensuring diagonality of the world-volume metric. Energy losses due to gravitational radiation are then computed in a standard way.

The total radiation power is presented as a sum over the harmonics of the basic frequency. The angular distribution of radiation at each harmonic is computed analyticaly in terms of Bessel functions. The spectrum is falling with the frequency, but there is a substantial contribution from high frequencies in the directions near the axis of symmetry. This part of radiation can be described in terms of Macdonald functions using well known approximation for the Bessel functions. Gravitational radiation vanishes in directions parallel to the axis, but the dominant contribution comes from directions close to the axis. Such a behavior may be attributed to the transversality of gravitational waves. The total radiation power is proportional to the total mass of the membrane, so the radiative timelife turns out to be independent of the membrane size. This is to be contrasted with the case of radiating string for which the timelife linearly grows with the size. Integration of the high frequency approximation gives a good estimate for the total radiation power.

2 The model

Consider a relativistic membrane $x^\mu = z^\mu(\sigma^a)$, $\mu = 0, 1, 2, 3$, interacting with the space-time metric $g_{\mu\nu}$ in four space-time dimensions (our signature convention is $(+\,-\,-\,-)$). The corresponding action

$$S = S_m + S_f \tag{1}$$

is the sum of the membrane term

$$S_m = \mu \int \sqrt{\gamma}\, d^3\sigma, \tag{2}$$

and the field term is

$$S_f = \frac{1}{16\pi G}\int \{-R\}\sqrt{-g}d^4x. \tag{3}$$

Here μ is the membrane tension parameter (of the dimension g/cm^2). The membrane world-volume is parameterized by the internal coordinates $\sigma^a = (\tau, \sigma, \rho)$, $\quad a = 0, 1, 2$, and

$$\gamma_{ab} = g_{\mu\nu}\frac{\partial z^\mu}{\partial \sigma^a}\frac{\partial z^\nu}{\partial \sigma^b} \tag{4}$$

is the induced metric, $\gamma = \det \gamma_{ab}$ (for more details on classical of membrans see, e.g. [10, 11]).

A variation of the action (2) over space-time coordinates z^μ gives the equation of motion

$$\frac{1}{\sqrt{\gamma}}\partial_a\left\{\sqrt{\gamma}\gamma^{ab}\partial_b\left(z^\nu g_{\mu\nu}\right)\right\} = 0, \tag{5}$$

where γ^{ab} is the inverse metric on the world-volume. Variation of the action (1) with respect to the space-time metric $g_{\mu\nu}$ gives the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{6}$$

with the membrane stress–tensor as a source

$$T^{\mu\nu} = \mu\int \sqrt{\gamma}\gamma^{ab}\partial_a z^\mu \partial_b z^\nu \frac{\delta^4\left(x - z(\sigma)\right)}{\sqrt{-g}}d^3\sigma \tag{7}$$

The action (1) is invariant under the reparameterization of the world-volume $\sigma^a \to \sigma^{a'}(\sigma^b)$, therefore three gauge conditions may be imposed on the world-volume metric γ_{ab}. In what follows we will consider the gravitational field perturbatively

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{8}$$

and impose on the induced world-volume metric γ_{ab} the flat space-time diagonality condition

$$\dot{z}^\mu z'_\mu = 0, \quad \dot{z}^\mu \bar{z}_\mu = 0, \quad z'^\mu \bar{z}_\mu = 0. \tag{9}$$

Here and below the contractions will be performed using the flat metric $\eta_{\mu\nu}$, and we denote $\dot{z}^\mu = \partial_\tau z^\mu$, $z'^\mu = \partial_\sigma z^\mu$, $\bar{z}^\mu = \partial_\rho z^\mu$. The induced metric then assumes the following form

$$\gamma_{ab} = \begin{pmatrix} \dot{z}^2 & 0 & 0 \\ 0 & z'^2 & 0 \\ 0 & 0 & \bar{z}^2 \end{pmatrix}, \quad \gamma = \sqrt{\dot{z}^2 z'^2 \bar{z}^2}, \tag{10}$$

where $\dot{z}^2 = \dot{z}^\mu \dot{z}_\mu$ etc. Note that $\dot{z}^\mu$ is a timelike vector, while z'^μ and $\bar{z}^\mu$ are spacelike, so the signature of γ_{ab} is $+--$, $\det\gamma_{ab}$ being positive. In the diagonal gauge the flat-space membrane equation of motion (5) can be presented in a simpler form as the continuity equation :

$$\frac{\partial P^\mu_\tau}{\partial \tau} + \frac{\partial P^\mu_\sigma}{\partial \sigma} + \frac{\partial P^\mu_\rho}{\partial \rho} = 0, \tag{11}$$

where P^μ_a are generalized momenta obtained by variating the membrane action over $\partial x_\mu/\partial\sigma^a$:

$$\begin{aligned} P^\mu_\tau &= \frac{1}{\sqrt{\gamma}}\dot{z}^\mu \mu z'^2 \bar{z}^2, \\ P^\mu_\sigma &= \frac{1}{\sqrt{\gamma}} z'^\mu \dot{z}^2 \bar{z}^2, \\ P^\mu_\rho &= \frac{1}{\sqrt{\gamma}}\bar{z}^\mu \dot{z}^2 z'^2. \end{aligned} \tag{12}$$

It is worth mentioning that, contrary to the case of the string, for the membrane one can not ensure the flat metric on the world-volume, therefore the equations (11) are still highly non-linear. They can be solved, however, assuming certain symmetric configurations to hold.

Finally, the linearized Einstein equations for the metric perturbations will read

$$\Box\psi^{\mu\nu} = 16\pi G T^{\mu\nu}, \tag{13}$$

where

$$\psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h^{\alpha}_{\alpha} \eta_{\mu\nu}, \tag{14}$$

and the membrane stress-tensor simplifies to

$$T^{\mu\nu} = \mu \int \left(\dot{z}^{\mu}\dot{z}^{\nu} z'^2 \bar{z}^2 + z'^{\mu} z'^{\nu} \dot{z}^2 \bar{z}^2 + \bar{z}^{\mu}\bar{z}^{\nu} \dot{z}^2 z'^2 \right) \\ \times \delta^4 \left(x - z(\sigma)\right) \left(\dot{z}^2 z'^2 \bar{z}^2\right)^{-1/2} d^3\sigma \quad . \tag{15}$$

3 Toroidal oscillating membrane

Consider a closed membrane whose world-volume is a product of a two-torus and a line. In order to be able to calculate the radiation power analytically we will assume that the only oscillation mode in excited, namely that corresponding to the smaller radius r of the torus. It is easy to see that the equations of motion (11) imply that this mode decouples from other modes in the limit when the bigger radius of the torus R is large with respect to r. Choosing the z-axis to be the axis of symmetry (with the large cicrle of the torus lying in the plane x, y) we arrive at the following parameterization of the membrane world-volume

$$\begin{aligned} z^0 &= \tau, \;\; z^1 = (R + r(\tau)\cos\rho)\cos\sigma, \\ z^2 &= (R + r(\tau)\cos\rho)\sin\sigma, \;\; z^3 = r(\tau)\sin\rho. \end{aligned} \tag{16}$$

This corresponds to a toroidal membrane of a constant large radius R a variable small one $r(\tau)$. The ansatz (16) gives an approximate solution of the equation of motion (11) valid if $R^2 \gg r^2$, which condition will be assumed so forth. In this approximation the world-volume element is equal to

$$\sqrt{\gamma} = R\sqrt{r^2\left(1 - \dot{r}^2\right)}. \tag{17}$$

Substituting this into the equations of motion (11), one finds that, under assumption that $R^2 \gg r^2$, the full system reduces to the single equation

$$\ddot{r}r + (1 - \dot{r}^2) = 0. \tag{18}$$

This is solved by the simple harmonic function

$$r = \Omega^{-1}\sin(\Omega\tau), \tag{19}$$

where Ω is a constant, and it is assumed for simplicity that $r(0) = 0$. The parameter Ω plays both the role of the oscillation frequency and the inverse radius of the smaller circle of the torus at the moments of maximal extention.

In our approximation each loop $\sigma = const$ belonging to the membrane moves indepentently and in exactly the same way as the corresponding circular closed string. Actually we deal with the transversal oscillations of a cylindric shape membrane which was bended to toroidal shape with the large second radius $R^2 \gg r^2$. Note that the function r in (16) may take both positive and negative values. Indeed, points on the loop moving radially inwards during contraction phase pass through the center and then continue their motion in the same direction, which now corresponds to an outward motion of the loop exercising an expansion phase. Clearly, the period of oscillations is π/Ω and not $2\pi/\Omega$, since the motion at negative r physically coincides with that at positive r.

4 Gravitational Radiation

The computation of gravitational radiation on the basis of the Einstein equations (13) is standard and amounts to the evaluation of the Fourrier-transform of the stress-tensor (15). Radiation power can be presented as the sum over the even harmonics of the basic frequency Ω (in view of the fact that the actual oscillation period is π/Ω):

$$P = \sum_{n=1}^{\infty} P_n, \quad \omega_n = 2n\Omega. \tag{20}$$

The angular distribution of radiation on the frequency ω_n for two polarizations is given by

$$\frac{dP_n}{dO} = \frac{\omega_n^2 G}{\pi} \sum_{\pm} |T^{\pm}(\mathbf{k},\, \omega_n)|^2 \tag{21}$$

where $\mathbf{k}$ is the wave vector, $\mathbf{k}^2 = \omega_n^2$, and

$$T^{\pm}(\mathbf{k},\, \omega_n) = \varepsilon^{\pm}_{\mu\nu} T^{\mu\nu}(\mathbf{k},\, \omega_n), \tag{22}$$

$$T^{\mu\nu}(\mathbf{k},\, \omega_n) = \frac{\Omega}{\pi} \int_0^{\pi} dt \int \mathrm{e}^{i(\omega_n t - \mathbf{kr})} T^{\mu\nu}(\mathbf{r},\, t) d^3x. \tag{23}$$

Here the vector $\mathbf{k}$ is

$$\mathbf{k} = \omega_n (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \tag{24}$$

We introduced chiral polarisation projectors

$$\begin{aligned} \varepsilon^{\pm}_{\mu\nu} &= \varepsilon^{\pm}_{\mu}\varepsilon^{\pm}_{\nu}, \\ \varepsilon^{\pm}_{\mu} &= \frac{1}{\sqrt{2}}(e^{(1)}_{\mu} \pm i e^{(2)}_{\mu}), \end{aligned} \tag{25}$$

where $e^{(1,2)}_{\mu}$ are two linearly independent space-like unit four-vectors orthogonal to the wave four-vector $k^{\mu} = (\omega_n, \mathbf{k})$ and its P-reflected dual $\bar{k}^{\mu} = (\omega_n, -\mathbf{k})$:

$$\begin{aligned} e^{(1,2)}_{\mu} k^{\mu} &= e^{(1,2)}_{\mu} \bar{k}^{\mu} = e^{(1)}_{\mu} e^{(2)\mu} = 0, \\ e^{(1)\mu} e^{(1)}_{\mu} &= e^{(2)\mu} e^{(2)}_{\mu} = -1. \end{aligned} \tag{26}$$

Alternatively, one can choose linear polarisation states decomposing

$$\varepsilon^{\pm}_{\mu\nu} = \frac{1}{\sqrt{2}} \left(e^{\oplus}_{\mu\nu} \pm i e^{\otimes}_{\mu\nu} \right). \tag{27}$$

Explicitly

$$\begin{aligned} e^{\oplus}_{\mu\nu} &= \frac{1}{\sqrt{2}} \left(e^{(1)}_{\mu} e^{(1)}_{\nu} - e^{(2)}_{\mu} e^{(2)}_{\nu} \right), \\ e^{\otimes}_{\mu\nu} &= \frac{1}{\sqrt{2}} \left(e^{(1)}_{\mu} e^{(2)}_{\nu} + e^{(2)}_{\mu} e^{(1)}_{\nu} \right) \end{aligned} \tag{28}$$

In the Lorentz frame where $e^{(1,2)}_0 = 0$, two unit vectors orthogonal to $\mathbf{k}$ can be parameterized as usual

$$\begin{aligned} \mathbf{e}^{(1)} &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta), \\ \mathbf{e}^{(2)} &= (-\sin\varphi, \cos\varphi, 0). \end{aligned} \tag{29}$$

Actual calculations give

$$T^{\pm}(\mathbf{k},\, \omega_n) = \frac{\mu R}{2\pi\Omega} \int_0^{\pi} d\chi \int_{-\pi}^{-\pi} d\rho \int_0^{2\pi} d\sigma \mathrm{e}^{2ni\psi} \Phi^{\pm}, \tag{30}$$

where ψ is

$$\psi = (\chi - \sin\rho\cos\theta\sin\chi - \Omega R\cos(\sigma - \varphi)\sin\theta)\,. \tag{31}$$

Note that the azimuthal angular variable φ enters into this expression (and all other formulas below) only in the combination $\sigma - \varphi$, and thus can be eliminated by the shift of the integration variable $\sigma \to \sigma - \varphi$. This shows that all quanitites are actually φ-independent, so without loss of generality we can put $\varphi = 0$:

$$\psi = (\chi - \sin\rho\cos\theta\sin\chi - \Omega R\cos\sigma\sin\theta)\,. \tag{32}$$

Then

$$\Phi^{\pm} = (\cos\sigma\cos\theta\cos\rho - \sin\theta\sin\rho \pm i\sin\sigma\cos\rho)^2 \tag{33}$$

The integrals can be evaluated in terms of the Bessel functions as follows. Consider first the integration over χ. One has

$$\int\limits_0^{\pi} \mathrm{e}^{2ni(\chi - z\sin\chi)} d\chi = \int\limits_0^{\pi} \cos\left[2n\left(\chi - z\sin\chi\right)\right] d\chi i \int\limits_0^{\pi} \sin\left[2n\left(\chi - z\sin\chi\right)\right] d\chi \pi J_{2n}(2nz) + iI \tag{34}$$

where $z = \sin\rho\cos\theta$. One can show that the integral over ρ of the quantity I vanishes. Indeed, after a constant shift of the integrationa variable $\chi \to \chi + \pi/2$ one obtains the following integral

$$\int\limits_{-\pi/2}^{\pi/2} d\chi \int\limits_{-\pi}^{\pi} d\rho \sin\left[2ni\left(\chi - z\cos\chi\right)\right], \tag{35}$$

where the integrand changes sign under the simultaneous inversion $\chi \to -\chi$, $\rho \to -\rho$.

Purely imaginary terms in $\Phi^{\pm}$ are proportional ether to $\sin 2\sigma$ or to $\sin 2\rho$; taking into account the explicit form of the phase factor ψ, one finds that both such contributions give zero because of the antisymmetry property of the integrand in appropriate variables. That means that gravitational radiation has only one linear polarisation component $\oplus$. Due to similar reasons, the cross product of the first two terms in (34) after integration over ρ also vanishes, so $\Phi^{\pm}$ can be replaced by the sum of squares of three terms, or after a rearrangement, by the following expression which is more suitable for further integration

$$\Phi = \frac{\sin^2\theta}{4}(1 - 3\cos 2\rho) + \frac{1 + \cos^2\theta}{4}\cos 2\sigma(1 + \cos 2\rho). \tag{36}$$

After integration, finally the resulting amplitude reads

$$\begin{aligned} T^{+}(\mathbf{k}, \omega_n) &= \tfrac{\pi^2\mu R}{2\Omega}\Big[(1 + \cos^2\theta)J_2(y)\left(J'^2_n - \tan^2\theta J_n^2\right) \\ &+ \sin^2\theta J_0(y)\left(J_n^2\left(1 + 3/\cos^2\theta\right) - 3J'^2_n\right)\Big], \end{aligned} \tag{37}$$

where the argument of Bessel functions of the order $n, n \pm 1$ is $x = n\cos\theta$. One can notice that the interference effects now are described by two different Bessel functions of the orders 0 and 2. Obviously this is related to the spin of the graviton. Since $J_2(0) = 0$, the first term in (37) vanishes at $\theta = 0, \pi$, while the second terms contains an explicit factor $\sin^2\theta$ and thus vanishes for these directions too. Thus, the amplitude of the gravitational radiation in directions collinear to the axis is zero.

For small $\sin\theta$ we find

$$\begin{aligned} J_n(na) &\approx \frac{\sin\theta}{\pi\sqrt{3}} K_{1/3}\left(\frac{n}{3}\sin^3\theta\right), \\ J'_n(na) &\approx \frac{\sin^2\theta}{\pi\sqrt{3}} K_{2/3}\left(\frac{n}{3}\sin^3\theta\right) \end{aligned} \tag{38}$$

Since Macdonald functions decrease exponentially for large arguments, it is clear that $\sin\theta$ is restricted by

$$\sin\theta < n^{-1/3}. \tag{39}$$

Now we observe that for $\sin\theta \sim n^{1/3}$, the argument of the Bessel function J_0 responsible for the interference effects is large $y \sim 2\Omega R n^{1/3} \gg 1$, so this becomes a rapidly oscillating factor. Thus, the smooth angular distribution described by Macdonald functions gets modulated by the interference ripples. One can average over ripples taking the asymptotics of J_0 at large arguments replacing the square of cosine by one half:

$$\overline{J_0^2(y)} \approx \frac{1}{\pi y}. \tag{40}$$

Note, that in the region of large y giving the main contribution to the radiated power one has $J_0 = -J_2$ (see (**??**), this can be taken into account before averaging the angular distribution over the ripples. Thus the angular distribution of radiation at n-th harmonics averaged over riples is given by

$$\frac{dP_n}{do} = \frac{4\mu^2 RnG}{9\pi^2\Omega} \sin^7\theta \left(3K_{1/3}^2(z) - K_{2/3}^2(z)\right)^2. \tag{41}$$

where

$$z = \frac{n}{3}\sin^3\theta. \tag{42}$$

Now we can integrate this distribution over all directions in view of the rapid convergence of the integrals of the Macdonald functions. Taking to the argument of the MacDonald functions as the integration variable and using the following relation for the remaining angular factor

$$\sin^8\theta d\theta = \frac{9}{n^3} z^2 dz. \tag{43}$$

(the additional factor $\sin\theta$ comes from do), one obtains

$$P = \frac{8\mu^2 RG}{\pi\Omega} I \sum_{n=1}^{\infty} \frac{1}{n^2}, \tag{44}$$

where

$$I = \int \left(3K_{1/3}^2 - K_{2/3}^2\right)^2 dz = 1.24, \tag{45}$$

and finally

$$P = \frac{4\pi}{3} I G\mu^2 \frac{R}{\Omega}. \tag{46}$$

For our configuration of membrane its area at the moments of maximal extension is $4\pi^2 R/\Omega$ (recall that Ω is equal to an inverse radius of the small circle of the torus, and we assumed that $R \gg 1/\Omega$), therefore, the quantity $\mu R/\Omega$ is proportional to the total mass. Thus, the membrane lifetime

$$\tau \sim \frac{\mu R/\Omega}{P} \sim \frac{1}{G\mu} \tag{47}$$

is independent on the membrane size, and depends only on a membrane tension. This should be contrasted to the case of the radiating string loops, whose lifetime is proportional to the loop size.

References

[1] R.A. Battye, R.R. Caldwell, and E.P.S. Shellard *Gravitational waves from cosmic strings. In "Topological defects in Cosmology", Eds. M.Sigmore and F.Melchiori, WS.Singapore*, 11-30 (1997)

[2] CJ. Burden *Gravitational radiation from a particular class cosmic strings. Phys.Lett.* **164B**,277-281 (1985)

[3] T.Vachaspati and A.Vilenkin Gravitational radiation from cosmic strings. *Phys.Rev* **D31** 3052 -3058 (1985)

[4] T.Vachaspati, A.Everett and A.Vilenkin Radiation from strings and domain walls. *Phys.Rev* **D30** 4525 -4535 (1992)

[5] A.Vilenkin and E.P.S.Shellard Cosmic strings and other topological defects. *CUP,Cambridge* (1992)

[6] ED.Copeland, M.Gleiser, and H.R.Miiler Oscillous:Resonant configulations during bubble formation. *Phys.Rev.* **D52**,1920-1933 (1995)

[7] M.Kamionowski, A.Kosowski, and M.Turner Gravitational radiation from first order phase transitions. *Phys.Rev* **D49** 2837-2851 (1994)

[8] M.Gleiser and R.Roberts Gravitational radiation from collapsing vacuum domains. *Phys.Rev.Lett* **81**5497-5500 (1998)

[9] A.Kosowski, M.Turner and R.Warkins Gravitational radiation from colliding vacuum bubbles. *Phys.Rev* **D45** 4514-4535 (1992)

[10] P.A.Collins and RW.Tucker Classical and quantum mechanic free relativistic membranes. *Nucl.Phys.* **B112** 150-176 (1976)

[11] A.Sugamoto Theory of membranes. *Nucl.Phys.* **B215** 381-406 (1983)