Black Hole Entropy from Poisson Brackets

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Abstract

It was suggested by S. Carlip that black hole entropy can be derived from a central charge of the Virasoro algebra arising as a subalgebra in the surface deformations of General Relativity in any dimension. Here it is argued that the argumentation based on Regge-Teitelboim's approach is unsatisfactory. The canonical generators are really "non-differentiable" and the standard Poisson brackets for these functionals are equal to zero so being unable to provide any Virasoro algebra with a central charge. Nevertheless Carlip's calculation will be correct if we admit another definition for the Poisson bracket. The new Poisson bracket differs from the standard one by surface terms and allows to work with "non-differentiable" functionals.

1 Introduction

The expresion for black hole entropy was obtained by S. Carlip in publication [1] on the base of the General Relativity canonical formalism and the algebra of hypersurface deformations. The Virasoro algebra with its central term arises as a result of the Poisson brackets calculation after taking into account boundary conditions on the black hole horizon. The similar argumentation based on application of the conformal field theory and Cardy formula was advocated also in the paper written by S. Solodukhin [2].

In [1] it is argued that Regge-Teitelboim's method based on a construction of the so-called "differentiable" generators is relevant. However, a detailed examination shows that this is really not the case. The Hamiltonian generators considered in paper [1] in fact are not "differentiable" under variations required to derive the Virasoro albebra. The straightforward and explicit calculation of the Poisson brackets according to the standard formula (based on the general derivation of the surface terms made by author for 3+1–space-time as early as in 1985 [3]) gives zero for the surface integral which should be the principal issue of paper [1].

We demonstrate here that in fact modified Poisson brackets, applicable in the more general case of "non-differentiable" functionals, are implicitly used in publication [1]. These new brackets differ from the standard ones in boundary terms and they have been used for the first time in paper [5]. Later they were generalized in two different ways [6], [7]. The difference between these two approaches does not manifest itself in the case under consideration. Recently the same brackets were independently used also in publication [8].

Therefore we observe that the derivation of the black hole entropy given in paper [1] formally stays valid if one changes the argumentation and declines the Regge-Teitelboim ideology in favor of the more general approach based on the new Poisson brackets.

2 Notations and Carlip's Calculation

We use the same notations as in publication [1]. So, the metric of n-dimensional (in further $n = 4$ case is preferred) space-time will be looking as follows

$$ds^2 = -N^2dt^2 + f^2(dr + N^r dt)^2 + \sigma_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt),$$

where $\alpha, \beta \ldots$ correspond to coordinates on a sphere $r = \text{const}$, $t = \text{const}$ and run over values $1, 2, \ldots, n-2$. The function N tends to zero on the horizon of a black hole $r = r_+$ in such a way that

$$N^2 = h(x^\alpha)(r - r_+) + O(r - r_+)^2. \tag{1}$$

Further, the Hamiltonian looks like a linear combination of constraints $\{\mathcal{H}_t, \mathcal{H}_a\}$ plus, probably, some surface integrals

$$H[\hat{\xi}] = \int_\Sigma d^{n-1}x\, \hat{\xi}^\mu \mathcal{H}_\mu, \qquad L[\hat{\xi}] = H[\hat{\xi}] + \oint_{\partial\Sigma} \ldots, \tag{2}$$

where the parameters of deformations of a constant time hypersurface (Lagrange multipliers for constraints) are components of the decomposition for an infinitesimal space-time diffeomorphism ξ^μ over the basis $\{(1/N, -N^a/N), \partial/\partial x^a\}$:

$$\hat{\xi}^t = N\xi^t, \qquad \hat{\xi}^a = \xi^a + N^a\xi^t. \tag{3}$$

If one calculate a variation over g_{ab} and π^{ab} of that part of a Hamiltonian containing only constraints (without the boundary terms), and consider ξ^μ as not depending of canonical variables, one get

$$\begin{aligned}\delta \quad & \int_\Sigma d^{n-1}x\,\hat{\xi}^\mu\mathcal{H}_\mu = \int_\Sigma d^{n-1}x\left(\frac{\delta H}{\delta g_{ab}}\delta g_{ab} + \frac{\delta H}{\delta \pi^{ab}}\delta\pi^{ab}\right) - \\ - \quad & \frac{1}{16\pi G}\oint_{\partial\Sigma} d^{n-2}x\left\{\sqrt{\sigma}\left(\sigma^{ac}n^b - \sigma^{ab}n^c\right)\left(\hat{\xi}^t\nabla_c\delta g_{ab} - \nabla_c\hat{\xi}^t\delta g_{ab}\right) + 2\hat{\xi}^a\delta\pi_a{}^r - \hat{\xi}^r\pi^{ab}\delta g_{ab}\right\},\end{aligned} \tag{4}$$

where n^a is a unit normal to the boundary at $t = \text{const}$, K_{ab} is an extrinsic curvature tensor of a constant time hypersurface, $\pi^{ab} = -f\sqrt{\sigma}(K^{ab} - g^{ab}K)$ are momenta conjugate to the spatial metric g_{ab} (the sign differs from [1]). The surface integral is taken over a boundary which should include, on the one hand, the horizon of a black hole, and on another – the spatial infinity. No boundary conditions are applied here. To within notations the formula coincides with the similar formula from publication by Regge and Teitelboim [4].

In further the game is entered by boundary conditions on the canonical variables and on the parameters of deformations. They are

1. to set in a phase space a region "close" to a black hole solution;

2. to provide conservation of this region under evolution assigned to the parameters of deformations.

Carlip starts with the following boundary conditions:

$$\begin{gathered} f = \frac{\beta h}{4\pi}N^{-1} + O(1), \qquad N^r = O(N^2), \qquad \sigma_{\alpha\beta} = O(1), \qquad N^\alpha = O(1), \\ (\partial_t - N^r\partial_r)g_{\mu\nu} = O(N)g_{\mu\nu}, \qquad \nabla_\alpha N_\beta + \nabla_\beta N_\alpha = O(N), \qquad \partial_r N = O(1/N), \\ K_{rr} = O(1/N^3), \qquad K_{\alpha r} = O(1/N), \qquad K_{\alpha\beta} = O(1), \\ \hat{\xi}^r = O(N^2), \qquad \hat{\xi}^t = O(N), \qquad \hat{\xi}^\alpha = O(1). \end{gathered}$$

What about a part of the boundary arranged at spatial infinity, by considering parameters $\hat{\xi}$ as rapidly decreasing we get rid of the corresponding contribution to the Hamiltonian variation.

Despite of the accepted boundary conditions the variation of Hamiltonian $H[\hat{\xi}]$ still contains some surface contribution and to get rid of it in publication [1] it is offered to add to a linear combination of constraints a surface integral of the following form

$$J[\hat{\xi}] = \frac{1}{8\pi G}\oint_{r=r_+} d^{n-2}x\left\{n^a\nabla_a\hat{\xi}^t\sqrt{\sigma} + \hat{\xi}^a\pi_a{}^r + \left[n_a\hat{\xi}^a K\sqrt{\sigma}\right]\right\}. \tag{5}$$

The last term here is not required in further, as the extra restrictions $K_{rr} = 0 = K_{\alpha\beta}$ are imposed on the exterior curvature tensor in [1]. It means that the last term in formula (4) is zero already and we in fact need not the term written in square brackets of (5). After that the variation of the improved Hamiltonian $L[\hat{\xi}] = H[\hat{\xi}] + J[\hat{\xi}]$ becomes

$$\delta L[\hat{\xi}] = \textit{bulk terms} + \frac{1}{8\pi G}\oint_{r=r_+} d^{n-2}x\left(\delta n^r\partial_r\hat{\xi}^t + \left[\frac{1}{f}\hat{\xi}^r\delta K_{rr} + \delta n_r\hat{\xi}^r K\right]\right)\sqrt{\sigma}. \tag{6}$$

The last two terms in further are not necessary as mentioned above.

Then in publication [1] it is stated that, first, we can calculate a Poisson bracket for generators of two various deformations according to the following formula

$$\left\{L[\hat{\xi}_2], L[\hat{\xi}_1]\right\} = \delta_{\hat{\xi}_2}L[\hat{\xi}_1], \tag{7}$$

And secondly, it is supposed, that this equality remain valid after reduction, that is, after taking constraints as zeros "in strong sense". The last means, that the formula remains valid for surface integrals taken separately:

$$\textit{boundary terms of } \left\{L[\hat{\xi}_2], L[\hat{\xi}_1]\right\} \approx \textit{boundary terms of } \delta_{\hat{\xi}_2} H[\hat{\xi}_1] + \delta_{\hat{\xi}_2} J[\hat{\xi}_1]. \tag{8}$$

As a result of such evaluation the following expression was obtained for the surface contribution to the Poisson bracket $\left\{L[\hat{\xi}_2], L[\hat{\xi}_1]\right\}$:

$$-\frac{1}{8\pi G}\oint_{r=r_+} d^{n-2}x\,\sqrt{\sigma}\left\{\frac{1}{f^2}\left[\partial_r(f\hat{\xi}_2^r)\partial_r\hat{\xi}_1^t - \partial_r(f\hat{\xi}_1^r)\partial_r\hat{\xi}_2^t\right] + \frac{1}{f}\partial_r\left[\hat{\xi}_1^r\partial_r\hat{\xi}_2^t - \delta_{\hat{\xi}_2}\hat{\xi}_1^t\right]\right\}. \tag{9}$$

In further it is used to realize the Virasoro algebra, which central charge has allowed to connect the number of horizon states with the well-known expression for the black hole entropy by Bekenstein-Hawking. Below we shall show, that the argumentation of paper [1] on a deduction of this formula should be reconsidered.

3 The Regge-Teitelboim Approach

In publication by Regge and Teitelboim [4], written in 1974, the Hamiltonian formalism of a field theory for the first time was applied to a problem, where the surface integrals originating in integration by parts were non-negligible, just opposite, they have an important physical meaning. Namely, after putting on gauges, that is, after reduction, the role of generators of the asymptotic Poincaré group (transformations that preserve boundary conditions of the asymptotically flat space - time) is played by the surface integrals. These surface integrals arise as a result of the Poisson brackets evaluation made according to the well-known formula

$$\{F + \oint_{\partial\Sigma}\ldots, G + \oint_{\partial\Sigma}\ldots\}_{\rm old} = \int_\Sigma d^{n-1}x\left(\frac{\delta F}{\delta q_A(x)}\frac{\delta G}{\delta p_A(x)} - \frac{\delta F}{\delta p_A(x)}\frac{\delta G}{\delta q_A(x)}\right),$$

and any modification of functionals F and G by surface integrals $\oint_{\partial\Sigma}\ldots$ (or, that is the same, a modification of their integrands by divergences) leaves this bracket untouched. It is obvious, as the standard bracket is defined with the help of Euler-Lagrange derivatives, and they are zero at any divergences. It is a common belief that the essence of Regge-Teitelboim's method is to fix surface terms in functionals in such a way that their variations look like

$$\delta F = \int_\Sigma d^{n-1}x\frac{\delta F}{\delta\phi_A(x)}\delta\phi_A(x),$$

without any boundary terms.

However, in the Regge-Teitelboim approach we can start also with the evaluation of surface integrals in Poisson brackets. This possibility was used in our publication [3]. Thus taking as initial Hamiltonians

$$H[\hat{\xi}] = \int_\Sigma d^{n-1}x\,\hat{\xi}^\mu\mathcal{H}_\mu \tag{10}$$

where $\{\mathcal{H}_t, \mathcal{H}_a\}$ are constraints, we obtain (we write below the corresponding formula from publication [3] in the notations of Carlip's work (here $n = 4$)) the following expression for their Poisson bracket:

$$\begin{aligned}\{H[\hat{\xi}_1] + \oint_{\partial\Sigma}\ldots, H[\hat{\xi}_2] + \oint_{\partial\Sigma}\ldots\}_{\rm old} = H[\hat{\xi}_3] + \frac{1}{8\pi G}\oint_{\partial\Sigma}\hat{\xi}_3^a\pi_a{}^b dS_b + \\ + \oint_{\partial\Sigma}\mathcal{H}_t(\hat{\xi}_1^t\hat{\xi}_2^a - \hat{\xi}_2^t\hat{\xi}_1^a)dS_a - \frac{1}{16\pi G}\oint_{\partial\Sigma}\pi^{ab}\left[\hat{\xi}_1^c(\hat{\xi}_{2a|b} + \hat{\xi}_{2b|a}) - \right. \\ \left. - \hat{\xi}_2^c(\hat{\xi}_{1a|b} + \hat{\xi}_{1b|a})\right]dS_c + \frac{1}{8\pi G}\oint_{\partial\Sigma}\sqrt{g}R_b^a\left(\hat{\xi}_1^t\hat{\xi}_2^b - \hat{\xi}_2^t\hat{\xi}_1^b\right)dS_a + \\ + \frac{1}{8\pi G}\oint_{\partial\Sigma}\sqrt{g}(g^{ab}g^{cd} - g^{ac}g^{bd})\left(\hat{\xi}_{1a}\hat{\xi}_{2|bd}^t - \hat{\xi}_{2a}\hat{\xi}_{1|bd}^t\right)dS_c,\end{aligned} \tag{11}$$

and

$$\hat{\xi}_3 = \{\hat{\xi}_1, \hat{\xi}_2\}_{\rm SD}, \qquad dS_a = f^{-1} n_a d^{n-2}x, \tag{12}$$

where

$$\begin{aligned} \{\hat{\xi}_1, \hat{\xi}_2\}^t_{\rm SD} &= \hat{\xi}^a_1 \partial_a \hat{\xi}^t_2 - \hat{\xi}^a_2 \partial_a \hat{\xi}^t_1 \\ \{\hat{\xi}_1, \hat{\xi}_2\}^a_{\rm SD} &= \hat{\xi}^b_1 \partial_b \hat{\xi}^a_2 - \hat{\xi}^b_2 \partial_b \hat{\xi}^a_1 + g^{ab}\left(\hat{\xi}^t_1 \partial_b \hat{\xi}^t_2 - \hat{\xi}^t_2 \partial_b \hat{\xi}^t_1\right). \end{aligned} \tag{13}$$

Now in the above we can take into account the boundary conditions from paper [1]. Then we get, for example,

$$\begin{aligned} &\frac{1}{8\pi G}\sqrt{g}(g^{ab}g^{cd} - g^{ac}g^{bd})\left(\hat{\xi}_{1a}\hat{\xi}^t_{2|bd} - \hat{\xi}_{2a}\hat{\xi}^t_{1|bd}\right) f^{-1} n_c = \\ &\qquad = \frac{1}{8\pi G} f\sqrt{\sigma}\sigma^{\alpha\beta}\left(\hat{\xi}^r_2 \hat{\xi}^t_{1|\alpha\beta} - \hat{\xi}^r_1 \hat{\xi}^t_{2|\alpha\beta}\right) = O(N^2), \end{aligned} \tag{14}$$

here we suppose $\hat{\xi}^\alpha = 0$ instead of $\hat{\xi}^\alpha = O(1)$ because this condition is used in Section 3 of [1]. In the similar way it occurs that all other integrands decrease as $O(N)$, or faster, i.e. they are zero on the black hole horizon. Therefore, the Regge-Teitelboim approach does not allow to derive expression (9) which is (2.13) of paper [1]. This is absolutely natural, because conditions for applying this method are just violated — variation δf, induced by a hypersurface deformation, does not fulfil Carlip's restriction $\delta f / f = O(N)$ and really has a form

$$\delta_{\hat{\xi}} f = \left(f\hat{\xi}^r\right)_{,r} = O(N^{-1}) \sim f,$$

that is, $\delta f / f = O(1)$. Due to this reason the surface integral from equation (2.10) of paper [1]

$$\delta L[\hat{\xi}] = \int_\Sigma d^{n-1}x \left(\frac{\delta H}{\delta g_{ab}}\delta g_{ab} + \frac{\delta H}{\delta \pi^{ab}}\delta\pi^{ab}\right) + \frac{1}{8\pi G}\oint_{r=r_+} d^{n-2}x\, \delta n^r \partial_r \hat{\xi}^t \sqrt{\sigma}. \tag{15}$$

is nonzero, that should be qualified in Regge-Teitelboim's terminology as "non-differentiability" of functional $L[\hat{\xi}] = H[\hat{\xi}] + J[\hat{\xi}]$, where

$$J[\hat{\xi}] = \frac{1}{8\pi G}\oint_{r=r_+} d^2x \left\{n^a \nabla_a \hat{\xi}^t \sqrt{\sigma} + \hat{\xi}^a \pi_a{}^r\right\}. \tag{16}$$

Here we omit term $n_a \hat{\xi}^a K\sqrt{\sigma}$, which is present in [1], because later in the consideration related to derivation of expression (2.13) Carlip supposes that $K_{rr} = 0 = K_{\alpha\beta}$ and ignores this term.

4 The New Poisson Brackets

There is a more general Hamiltonian approach where all functionals are admissible, including those giving nonzero boundary contribution to the first variation. It was for the first time shown in publication [5] that the standard Poisson bracket can be generalized by adding to it some divergence terms. As a result, the bracket so generalized can generate variations with nonzero surface contribution. This is just required in the case under consideration when the surface is a black hole horizon. Let us mention that two different extensions of the formula proposed in [5] were suggested later. They can be important in cases when surface contributions to functional variations contain arbitrary spatial derivatives for variations of fields and their conjugate momenta (generally speaking, arbitrary non-canonical variables): [6], [7]. An attempt of comparison of these two different formulas was done in paper [9], but here all give one and the same result.

Let us explain this in more detail. Let field variables are not necessarily canonical, but their Poisson bracket is ultralocal

$$\{\phi_A(x), \phi_B(y)\} = I_{AB}\delta(x, y),$$

then the standard bracket of two local functionals has the following form

$$\{F,G\} = \int_\Sigma d^{n-1}x \frac{\delta F}{\delta\phi_A(x)} I_{AB} \frac{\delta G}{\delta\phi_B(x)},$$

and for a "differentiable" functional we have

$$\delta F = \int_\Sigma d^{n-1}x \frac{\delta F}{\delta\phi_A(x)} \delta\phi_A(x).$$

Let now deal with "non-differentiable" functional in Regge-Teitelboim's sense, and let its first variation has the simplest possible form with a surface term

$$\delta F = \int_\Sigma d^{n-1}x \frac{\delta^\wedge F}{\delta\phi_A(x)} \delta\phi_A(x) + \oint_{\partial\Sigma} d^{n-2}x \frac{\delta^\vee F}{\delta\phi_A(x)} \delta\phi_A(x).$$

Then we can formally construct a full variational derivative as follows

$$\frac{\delta F}{\delta\phi_A(x)} = \theta(\Sigma)\frac{\delta^\wedge F}{\delta\phi_A(x)} + \delta(\partial\Sigma)\frac{\delta^\vee F}{\delta\phi_A(x)},$$

to include a surface (or boundary) contribution (here $\delta(\partial\Sigma)$ is the Dirac delta-function concentrated on the boundary of the integration region Σ and $\theta(\Sigma)$ is the Heaviside step function equal to 1 inside Σ and to 0 outside of it). So, we treat the full variational derivative as follows

$$\int_{\mathbb{R}^{n-1}} d^{n-1}x \frac{\delta F}{\delta\phi_A(x)} \delta\phi_A(x) \equiv \int_\Sigma d^{n-1}x \frac{\delta^\wedge F}{\delta\phi_A(x)} \delta\phi_A(x) + \oint_{\partial\Sigma} d^{n-2}x \frac{\delta^\vee F}{\delta\phi_A(x)} \delta\phi_A(x).$$

It is possible to put these full variational derivatives into the standard formula for the Poisson bracket (4) instead of the usual Euler-Lagrange derivatives. Then we get

$$\{F,G\}_{\text{new}} = \{F,G\}_{\text{old}} + \oint_{\partial\Sigma} d^{n-2}x \frac{\delta^\vee F}{\delta\phi_A} \delta_G\phi_A - \oint_{\partial\Sigma} d^{n-2}x \frac{\delta^\vee G}{\delta\phi_A} \delta_F\phi_A + ?!,$$

where

$$\delta_F\phi_A = I_{AB}\frac{\delta^\wedge F}{\delta\phi_B} = \{\phi_A, F\}, \qquad \delta_G\phi_A = I_{AB}\frac{\delta^\wedge G}{\delta\phi_B} = \{\phi_A, G\},$$

and ?! denotes the puzzling term corresponding to the delta-function squared. In publication [5] it was demanded to kill this term by a boundary condition which guarantees zero coefficient before the dangerous expression $[\delta(\partial\Sigma)]^2$:

$$\left(\frac{\delta^\vee F}{\delta\phi_A(x)} I_{AB} \frac{\delta^\vee G}{\delta\phi_B(x)}\right)_{\partial\Sigma} = 0.$$

Further attempts to extend the result of publication [5] were gone in two directions:

1. searching for regular expressions corresponding to terms like ?! [6];
2. postulating that these terms are absent in the final formula [7].

Therefore, the contribution to expression (9) (which is (2.13) of paper [1]) arises just due to "non-differentiability" of functional $L[\hat{\xi}]$

$$\delta L[\hat{\xi}] = \int_\Sigma d^{n-1}x \left(\frac{\delta H}{\delta g_{ab}}\delta g_{ab} + \frac{\delta H}{\delta \pi^{ab}}\delta\pi^{ab}\right) - \frac{1}{8\pi G}\oint_{r=r_+} d^2x\sqrt{\sigma}\frac{\delta f}{f^2}\hat{\xi}^t_{,r} \neq 0.$$

The variation $\delta_{\hat{\xi}} f$ is easily determined from the equations of motion generated by the Hamiltonian $L[\hat{\xi}]$

$$\delta_{\hat{\xi}} f = \{f, L[\hat{\xi}]\} = \dot{\sqrt{g_{rr}}} = \frac{\dot{\hat{\xi}}_{r|r}}{f} = f\left(\hat{\xi}^r_{,r} + \Gamma^r_{rr}\hat{\xi}^r\right) = \left(f\hat{\xi}^r\right)_{,r}. \tag{17}$$

By this way we get the first surface contribution to the Poisson bracket $\{L[\hat{\xi}_1], L[\hat{\xi}_2]\}_{\rm new}$ standing in expression (9)

$$-\frac{1}{8\pi G}\oint_{r=r_+} d^2x\frac{\sqrt{\sigma}}{f^2}\left[\delta_{\hat{\xi}_2} f\hat{\xi}^t_{1,r} - \delta_{\hat{\xi}_1} f\hat{\xi}^t_{2,r}\right] =$$

$$= -\frac{1}{8\pi G}\oint_{r=r_+} d^2x\frac{\sqrt{\sigma}}{f^2}\left[\left(f\hat{\xi}^r_2\right)_{,r}\hat{\xi}^t_{1,r} - \left(f\hat{\xi}^r_1\right)_{,r}\hat{\xi}^t_{2,r}\right]. \qquad (18)$$

The genesis of the second term of expression (9)

$$-\frac{1}{8\pi G}\oint_{r=r_+} d^2x\frac{\sqrt{\sigma}}{f}\partial_r\left[\delta_{\hat{\xi}_1}\hat{\xi}^t_2 - \delta_{\hat{\xi}_2}\hat{\xi}^t_1\right] = -\frac{1}{8\pi G}\oint_{r=r_+} d^2x\frac{\sqrt{\sigma}}{f}\partial_r\left[\hat{\xi}^r_1\partial_r\hat{\xi}^t_2 - \hat{\xi}^r_2\partial_r\hat{\xi}^t_1\right], \qquad (19)$$

is connected with the fact that the deformation parameters $\hat{\xi}^t$ themselves are dependent on the canonical variables and so themselves have nonzero Poisson brackets with the Hamiltonian. When these parameters come as multipliers before constraints this fact leads simply to changes in the final constraint multipliers and it is not important for us as we are on the constraint surface. But when they come in the nonzero surface integrals their variations $\delta_{\hat{\xi}_2}\hat{\xi}^t_1 = \hat{\xi}^a_2\partial_a\hat{\xi}^t_1$ will give a contribution to the Poisson bracket.

5 Conclusion

We have shown that the validity of Carlip's results is in fact based on the application of the new formula for Poisson brackets. This is unfortunately not clear from the argumentation given in [1]. Formulas (7) (which is (2.11) of [1]) and (8) are not valid if the Poisson bracket is defined by the standard expression. So, all this testifies in favor of the more general approach to the boundary problems in Hamiltonian dynamics [6], [10], [11].

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