Quantum Theory and Nonhamiltonian Systems. Path-Integral Approach

V.E. Tarasov
Institute of Nuclear Physics, Moscow State University, Moscow, RUSSIA.
tarasov@theory.npi.msu.su

In this talk we discuss the path-integral approach to quantum nonhamiltonian quantum systems.

1 Evolution of state

An important property of most nonhamiltonian systems is the entropy variation. Nevertheless, the quantum evolution of state (von Neumann equation)

$$\frac{\partial}{\partial t}\varrho_s(t) \; = -\frac{i}{\hbar}[H, \varrho_s(t)] \tag{1}$$

preserves the entropy $< S >= -Sp[\varrho_s(t) ln \varrho_s(t)]$ unchanged. Therefore, to describe quantum nonhamiltonian and dissipative systems, one normally uses a generalization of (1). Different authors have examined different forms of the generalizations proposed for (1). However, these generalizations are not related to the Liouville equation for classical nonhamiltonian system. In the Liouville picture the density function $\varrho(p,q,t)$ for classical nonhamiltonian system evolves according to the Liouville equation

$$\frac{d}{dt}\varrho(p,q,t) = -\mathcal{L}\varrho(p,q,t) - \Omega(p,q,t)\varrho(p,q,t) \ , \tag{2}$$

$$\Omega(p,q,t) \equiv -\sum_{k=1}^{n}\Big(\mathcal{L}(\{q^k,p^k\}) - \{\mathcal{L}(q^k),p^k\} - \{q^k,\mathcal{L}(p^k)\}\Big) \ .$$

The states of a quantum nonhamiltonian system at time t are described by the density operator $\varrho_s(t)$. The quantum analog of the Liouville equation (2) has the form [1]

$$\frac{d}{dt}\varrho_s(t) = -\mathcal{L}_s(\varrho_s(t)) - \frac{1}{2}(\Omega_s(t)\varrho_s(t) + \varrho_s(t)\Omega_s(t)) \ , \tag{3}$$

$$\Omega_s(t) = \frac{i}{\hbar}\sum_{k=1}^{n} J_{\mathcal{L}_s}(Q_k,P_k) \ , \quad J_{\mathcal{L}_s}(Q_k,P_k) \equiv \mathcal{L}_s([Q_k,P_k]) - [\mathcal{L}_s(Q_k),P_k] - [Q_k,\mathcal{L}_s(P_k)] \tag{4}$$

The quantum system whose evolution is given by equation(3) is called a quantum nonhamiltonian system if at least one of the Jacobians

$$J_{\mathcal{L}_s}(X_k,X_l) \equiv \mathcal{L}_s([X_k,X_l]) - [\mathcal{L}_s(X_k),X_l] - [X_k,\mathcal{L}_s(X_l)]$$

differs from zero, where $X_k = Q_k, P_k$. The quantum nonhamiltonian system whose evolution is given by equation(3) is called a quantum dissipative system if operator $\Omega_s(t)$ can be represented as (4) and differs from zero.

Let us define superoperators Ω^l_t, Ω^l_t by $\Omega^l_t A = \Omega_s(t)A$, $\Omega^r_t A = A\Omega_s(t)$, and the Liouville superoperator

$$\Lambda_t = -\mathcal{L}_s - \frac{1}{2}(\Omega^l_t + \Omega^r_t) \ .$$

The Liouville-von Neumann equation (3) can be represented as the linear equation

$$\frac{d}{dt}\varrho_s(t) = \Lambda_t\varrho_s(t) \ . \tag{5}$$

This equation has a formal solution

$$\varrho_s(t) = S(t,t_0)\varrho_s(t_0) \ , \quad S(t,t_0) = T\{exp\int_{t_0}^{t} d\tau\Lambda_\tau\} \ , \tag{6}$$

which $S(t,t_0)$ being the evolution superoperator. If the Liouville superoperator Λ_t is time-independent the evolution superoperator $S(t,t_0)$ is given by $S(t,t_0) = exp(t-t_0)\Lambda$.

We can define normalized density operator $\rho_s(t) \equiv \varrho_s(t)/Sp\varrho_s(t)$. The evolution equation can be written in the form [2]

$$\frac{d}{dt}\rho_s(t) = \Lambda_t\rho_s(t) - \rho_s(t)Sp[\Lambda_t\rho_s(t)]$$

which is nonlinear.

2 Path integral

In the coordinate representation the kernel $\varrho_s(q,q',t) =< q|\varrho_s(t)|q' >$ of the density operator $\varrho_s(t)$ evolves according to the equation

$$\varrho_s(q,q',t) = \int dq_0 dq'_0 S(q,q',q_0,q'_0,t,t_0)\varrho_s(q_0,q'_0,t) \ .$$

The kernel $S(q,q',q_0,q'_0,t,t_0)$ of the evolution superoperator $S(t,t_0)$ is called a probability amplitude. Let the Liouville superoperator Λ_t be time-independent, i.e. the evolution superoperator $S(t,t_0)$ is given by $S(t,t_0) = exp(t-t_0)\Lambda$.

The probability amplitude can be written as path integral [2]:

$$S(q,q',q_0,q'_0,t,t_0) = \int \mathcal{D}q\mathcal{D}q'\mathcal{D}p\mathcal{D}p' exp\int_{t_0}^{t} dt\Big(\frac{\imath}{\hbar}[\dot{q}p - \dot{q}'p'] + \Lambda(q,p|p',q')\Big) \ . \qquad (7)$$

This form is the integral over all trajectories in the phase space with the constraints that $q(t_0) = q_0$, $q(t) = q$, $q'(t_0) = q'_0$, $q'(t) = q'$ and the measure

$$\mathcal{D}q = \prod t dq(t) \ , \quad \mathcal{D}p = \prod_t \frac{dp(t)}{2\pi\hbar} \ .$$

The symbol $\Lambda(q,p|q',p')$ of Liouville superoperator Λ is defined by

$$\Lambda(q,y|y',q') = \frac{1}{(2\pi\hbar)^{2n}}\int dpdp'\Lambda(q,p|p',q')exp\frac{\imath}{\hbar}[(q-y)p - (q'-y')p'] \ .$$

$$\Lambda(q,y|y',q') \equiv \Lambda(q,q',y,y') =< q|\Lambda(\hat{P}(y,y'))|q' > \ , \quad \hat{P}(y,y') = |y><y'| \ .$$

3 Quantum Markovian equation

Lindblad [3] has shown that there exists a one-to-one correspondence between the completely positive norm continuous semigroup of superoperators $S(t,t_0)$ and completely dissipative superoperator Λ. The structural theorem of Lindblad gives the most general form of bounded completely dissipative Liouville superoperator. The Liouville-von Neumann equation for Lindbland system (Markovian master equation) in the Schrodinger picture is

$$\frac{d\rho_s(t)}{dt} = -\frac{i}{\hbar}[H,\rho_s(t)] + \frac{1}{2\hbar}\sum_{k=1}^{m}([V_k\rho_s(t),V_k^*] + [V_k,\rho_s(t)V_k^*]) \ . \qquad (8)$$

The basic assumption is that the general form (8) of a bounded completely dissipative superoperator Λ (8) given by the Lindblad theorem is also valid for an unbounded completely dissipative superoperator. Another condition imposed on the operators H, V_k, V_k^* is that they are functions of the observables of the one-dimensional quantum system P Q (with $[Q,P] = i\hbar I$). Let us consider $V_k = a_kP + b_kQ$, $k = 1,2$, where a_k, b_k are complex numbers, and the Hamilton operator H given by

$$H = H_0 + \frac{\mu}{2}(PQ + QP) \ , \quad H_0 = \frac{1}{2m}P^2 + U(Q) \ , \qquad (9)$$

Then with the notation [4]:

$$d_{qq} = \frac{\hbar}{2} \sum_{k=1,2} |a_k|^2 \ , \quad d_{pp} = \frac{\hbar}{2} \sum_{k=1,2} |b_k|^2 \ , \quad d_{pq} = -\frac{\hbar}{2} Re(\sum_{k=1,2} a_k^* b_k) \ , \quad \lambda = -Im(\sum_{k=1,2} a_k^* b_k)$$

the equation (8) takes the following form:

$$\frac{d\rho_s(t)}{dt} = -\frac{i}{\hbar}[H_0, \rho_s(t)] + \frac{i(\lambda-\mu)}{2\hbar}[P, \rho_s(t)Q + Q\rho_s(t)] - \frac{i(\lambda+\mu)}{2\hbar}[Q, \rho_s(t)P + P\rho_s(t)] -$$

$$-\frac{d_{pp}}{\hbar^2}[Q,[Q,\rho_s(t)]] - \frac{d_{qq}}{\hbar^2}[P,[P,\rho_s(t)]] + \frac{2d_{pq}}{\hbar^2}[P,[Q,\rho_s(t)]] \ . \tag{10}$$

The equation (10) is called Dekker model [5] which is phenomenological dissipative model. With an appropriate choice of parameters for Dekker model, the Caldeira-Leggett master equation [6] can be reproduced.

4 Path integral for Lindblad system

The Liouville superoperator for the Lindblad system (8) can be written as

$$\Lambda = -\frac{\imath}{\hbar}(H^l - H^r) + \frac{1}{2\hbar} \sum_{k=1}^{m} ((V_k^{*r} - V_k^{*l})V_k^l + (V_k^l - V_k^r)V_k^{*l} \ .$$

The kernel of the superoperator Λ is the kernel of operator $\Lambda(\hat{P}(y,y'))$:

$$\Lambda(q,y|y',q') = \Lambda(q,q',y,y') =< q|\Lambda(\hat{P}(y,y'))|q' >=$$

$$= -\frac{\imath}{\hbar}(< y'|q' >< q|H|y > - < q|y >< y'|H|q' >) + \frac{1}{\hbar} \sum_{k=1}^{m} < q|V_k|y >< y'|V_k^*|q' > -$$

$$-\frac{1}{2\hbar} \sum_{k=1}^{m} (< y'|q' >< q|V_k^* V_k|y > - < q|y >< y'|V_k^* V_k|q' >) \ .$$

The symbol $\Lambda(q,p|q',p')$ of Liouville superoperator Λ can be derived as

$$\Lambda(q,y|y',q') = \int dpdp' \{ -\frac{\imath}{\hbar}(< y'|p' >< p'|q' >< q|H_1|p >< p|y > -$$

$$- < q|p >< p|y >< y'|p' >< p'|H_2|q' >) + \frac{1}{\hbar} \sum_{k=1}^{m} < q|V_k|p >< p|y >< y'|p' >< p'|V_k^*|q' >) \} \ ,$$

with

$$H_1 \equiv H - \frac{\imath}{2} \sum_{k=1}^{m} V_k^* V_k \ , \quad H_2 \equiv H + \frac{\imath}{2} \sum_{k=1}^{m} V_k^* V_k \ , \quad H_2 = H_1^*$$

Then the symbol of Liouville superoperator Λ for the quantum Lindblad systems can be written in the form

$$\Lambda(q,p|p',q') = -\frac{\imath}{\hbar}[H_1(q,p) - H_2(p',q') + i \sum_{k=1}^{m} V_k(q,p) V_k^*(p',q')] \ ,$$

$$\Lambda(q,p|p',q') = -\frac{\imath}{\hbar}[H(q,p) - H(p',q')] - \frac{1}{2\hbar} \sum_{k=1}^{m} [(V_k^* V_k)(q,p) + (V_k^* V_k)(p',q') - 2V_k(q,p)V_k^*(p',q')] \ .$$

In the hamiltonian case ($V_k = 0$) the symbol is given by

$$\Lambda(q,p|p',q') = -\frac{\imath}{\hbar}[H(q,p) - H(p',q')] \ .$$

The probability amplitude for a quantum Lindblad system has the following form:

$$S(q,q',q_0,{q'}_0,t,t_0)=\int \mathcal{D}q\mathcal{D}p\mathcal{D}q'\mathcal{D}p'\ \mathcal{F}(q,q',p,p')\ e^{\frac{i}{\hbar}(\mathcal{A}(q,p)-\mathcal{A}(p',q'))}\ .$$

The following action functionals are used

$$\mathcal{A}(q,p)=\int_{t_0}^{t} dt(\dot{q}p-H(q,p))\ ,\quad \mathcal{A}(p',q')=\int_{t_0}^{t} dt(\dot{q}'p'-H(p',q'))\ .$$

The functional $\mathcal{F}(q,q',p,p')$ describes the nonhamiltonian part of evolution and is called an influence functional. The Lindblad system (8) is described by the functional

$$\mathcal{F}(q,q',p,p')=exp-\frac{1}{2\hbar}\int_{t_0}^{t} dt\sum_{k=1}^{m}[(V_k^*V_k)(q,p)+(V_k^*V_k)(p',q')-2V_k(q,p)V_k^*(p',q')]\ .$$

For the systems corresponding to the equation (10) it becomes

$$\mathcal{F}(q,q',p,p')=exp\frac{1}{\hbar^2}\int_{t_0}^{t} dt[2d_{qp}(q-q')(p-p')-d_{qq}(p-p')^2-d_{pp}(q-q')^2+$$

$$+i\hbar\lambda(pq'-qp')+i\hbar\mu(q'p'-qp)]\ .$$

The path integral (7) can be used for the quantum non-Markov and post-Markov master equations [7, 8] and to the Ghirardi model [9] for spontaneous reduction of the wave function.

It is very interesting to give a path integral formulation of nonhamiltonian classical mechanics by generalization of Gozzi-Reuter-Thaker integral [10]-[12].

References

[1] *V. E. Tarasov*, Theor. Math. Phys. 110 (1997) 57; Proceedings of the XI International Workshop on High Energy Physics and Quantum Field Theory. Moscow: MSU, 1997. p.368.
[2] *V. E. Tarasov*, Mathematical Introduction to Quantum Mechanics. Moscow: MAI, 2000.
[3] *G. Lindblad*, Commum. Math. Phys. 48 (1976) 119.
[4] *A. Sandulescu, H. Scutaru*, Ann. Phys. 173 (1983) 277.
[5] *H. Dekker*, Phys. Rep. 80 (1981) 1.
[6] *A. O. Caldeira, A. J. Leggett*, Physica A 121 (1983) 587.
[7] *T. Yu, L. Diosi, N. Gisin, W. T. Strunz*, Preprint quant-ph/9905006.
[8] *L. Diosi, N. Gisin, W. T. Strunz*, Phys. Rev. A. 58 (1998) 1699.
[9] *G. C. Chirardi, R. Grassi, F. Benatti*, Found. Phys. 22 (1995) 5; Workshop Quantum Dissipation and Application. (Trieste, Italy, 1996) SMR. 962/3.
[10] *E. Gozzi*, Preprint NBI-TH 87/77; Phys. Lett. B. 201 (1988) 525.
[11] *E. Gozzi, M. Reuter*, Preprint CERN-TH 5586/89.
[12] *E. Gozzi, M. Reuter, W. D. Thacker*, Phys. Rev. D. 40 (1989) 3363; D 46 (1992) 757.