In[2]:=
```
<< MB/MB.m
```

```
MB 1.1

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more info in hep-ph/0511200

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```

```
(*   The integrand of the MB integral for the one-loop massless box diagram with  p1^2=
 p2^2=p3^2=p4^2=0                      *)
```

In[3]:=
```
Box1[a1_, a2_, a3_, a4_] :=
  (S^(2-a1-a2-a3-a4-ep-z) T^z Gamma[a1 + a2 + a3 + a4 - 2 + ep + z] Gamma[a2 + z] Gamma[a4 + z]
     Gamma[2 - a1 - a2 - a4 - ep - z] Gamma[2 - a2 - a3 - a4 - ep - z] Gamma[-z] ) /
   (Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[4 - a1 - a2 - a3 - a4 - 2 ep]);
```

```
(*  Notation:
  s=(p1+p2)^2=-S, t=(p1+p2)^2=-T;
I Pi^(d/2) is pulled out, as always  *)
```

```
(*   The box with the powers of the propagators equal to one *)
```

In[4]:=
```
Box1[1, 1, 1, 1]
```

Out[4]=
$$\frac{S^{-2-ep-z}\, T^{z}\, \text{Gamma}[-1-ep-z]^{2}\, \text{Gamma}[-z]\, \text{Gamma}[1+z]^{2}\, \text{Gamma}[2+ep+z]}{\text{Gamma}[-2\, ep]}$$

In[5]:=
```
B1 = % /. {S → 1, T → x}
```

Out[5]=
$$\frac{x^{z}\, \text{Gamma}[-1-ep-z]^{2}\, \text{Gamma}[-z]\, \text{Gamma}[1+z]^{2}\, \text{Gamma}[2+ep+z]}{\text{Gamma}[-2\, ep]}$$

In[6]:=
```
B1Rules = MBoptimizedRules[B1, ep → 0, {}, {ep}]
```

MBrules::norules :  no rules could be found to regulate this integral

Out[6]=
$$\left\{\{ep \to -1\},\ \left\{z \to -\frac{1}{2}\right\}\right\}$$

In[7]:=
```
B1cont = MBcontinue[B1, ep → 0, B1Rules]
```

```
Level 1

Taking -residue in z = -1 - ep

Level 2

Integral {1}

2 integral(s) found
```

Out[7]=
$$\Bigg\{\Bigg\{\text{MBint}\Bigg[-\frac{\text{EulerGamma}\, x^{-1-ep}\, \text{Gamma}[-ep]^{2}\, \text{Gamma}[1+ep]}{\text{Gamma}[-2\, ep]} - \frac{x^{-1-ep}\, \text{Gamma}[-ep]^{2}\, \text{Gamma}[1+ep]\, \text{Log}[x]}{\text{Gamma}[-2\, ep]} - \frac{2\, x^{-1-ep}\, \text{Gamma}[-ep]^{2}\, \text{Gamma}[1+ep]\, \text{PolyGamma}[0, -ep]}{\text{Gamma}[-2\, ep]} + \frac{x^{-1-ep}\, \text{Gamma}[-ep]^{2}\, \text{Gamma}[1+ep]\, \text{PolyGamma}[0, 1+ep]}{\text{Gamma}[-2\, ep]},\ \{\{ep \to 0\},\ \{\}\}\Bigg]\Bigg\},$$
$$\text{MBint}\Bigg[\frac{x^{z}\, \text{Gamma}[-1-ep-z]^{2}\, \text{Gamma}[-z]\, \text{Gamma}[1+z]^{2}\, \text{Gamma}[2+ep+z]}{\text{Gamma}[-2\, ep]},\ \left\{\{ep \to 0\},\ \left\{z \to -\frac{1}{2}\right\}\right\}\Bigg]\Bigg\}$$

In[8]:= **B1select = MBpreselect[B1cont, {ep, 0, 0}]**

Out[8]= $\left\{\text{MBint}\left[-\frac{\text{EulerGamma}\, x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]}{\text{Gamma}[-2\,ep]} - \frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{Log}[x]}{\text{Gamma}[-2\,ep]} - \frac{2\,x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,-ep]}{\text{Gamma}[-2\,ep]} + \frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,1+ep]}{\text{Gamma}[-2\,ep]}, \{\{ep\to 0\},\{\}\}\right]\right\}$

In[9]:= **B1select = MBpreselect[B1cont, {ep, 0, 1}]**

Out[9]= $\left\{\text{MBint}\left[-\frac{\text{EulerGamma}\, x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]}{\text{Gamma}[-2\,ep]} - \frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{Log}[x]}{\text{Gamma}[-2\,ep]} - \frac{2\,x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,-ep]}{\text{Gamma}[-2\,ep]} + \frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,1+ep]}{\text{Gamma}[-2\,ep]}, \{\{ep\to 0\},\{\}\}\right], \text{MBint}\left[\frac{x^z\,\text{Gamma}[-1-ep-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+ep+z]}{\text{Gamma}[-2\,ep]}, \left\{\{ep\to 0\}, \left\{z\to -\frac{1}{2}\right\}\right\}\right]\right\}$

In[10]:= **B1exp = MBexpand[B1select, E^(EulerGamma ep), {ep, 0, 1}]**

Out[10]= $\left\{\text{MBint}\left[\frac{4}{ep^2\,x} - \frac{4\,\pi^2}{3\,x} - \frac{2\,\text{Log}[x]}{ep\,x} + \frac{7\,ep\,\pi^2\,\text{Log}[x]}{6\,x} + \frac{ep\,\text{Log}[x]^3}{3\,x} + \frac{17\,ep\,\text{PolyGamma}[2,1]}{3\,x}, \{\{ep\to 0\},\{\}\}\right], \text{MBint}\left[-2\,ep\,x^z\,\text{Gamma}[-1-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+z], \left\{\{ep\to 0\}, \left\{z\to -\frac{1}{2}\right\}\right\}\right]\right\}$

In[11]:= **res1 = B1exp[[1]][[1]]**

Out[11]= $\frac{4}{ep^2\,x} - \frac{4\,\pi^2}{3\,x} - \frac{2\,\text{Log}[x]}{ep\,x} + \frac{7\,ep\,\pi^2\,\text{Log}[x]}{6\,x} + \frac{ep\,\text{Log}[x]^3}{3\,x} + \frac{17\,ep\,\text{PolyGamma}[2,1]}{3\,x}$

In[12]:= **B1exp[[2]]**

Out[12]= $\text{MBint}\left[-2\,ep\,x^z\,\text{Gamma}[-1-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+z], \left\{\{ep\to 0\}, \left\{z\to -\frac{1}{2}\right\}\right\}\right]$

In[13]:= **int1 = B1exp[[2]][[1]]**

Out[13]= $-2\,ep\,x^z\,\text{Gamma}[-1-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+z]$

In[14]:= **Simplify[% //. { Gamma[-1 - z] → Gamma[-z] / (-1 - z), Gamma[2 + z] → Gamma[1 + z] (1 + z) }]**

Out[14]= $-\frac{2\,ep\,x^z\,\text{Gamma}[-z]^3\,\text{Gamma}[1+z]^3}{1+z}$

In[15]:= **% /. {Gamma[-z]^3 Gamma[1 + z]^3 → -π^3 Csc[π z]^3}**

Out[15]= $\frac{2\,ep\,\pi^3\,x^z\,\text{Csc}[\pi\,z]^3}{1+z}$

**(*  Now we take residues at z=0,1,2,.,..   *)**

In[16]:= **% /. z → z + n**

Out[16]= $\frac{2\,ep\,\pi^3\,x^{n+z}\,\text{Csc}[\pi\,(n+z)]^3}{1+n+z}$

In[17]:= `% /. {Csc[π (n + z)] → (-1)^n Csc[π z] }`

Out[17]= $\frac{2 (-1)^{3n} \text{ep} \pi^3 x^{n+z} \text{Csc}[\pi z]^3}{1+n+z}$

In[18]:= `% /. { (-1)^(3 n) → (-1)^n}`

Out[18]= $\frac{2 (-1)^{n} \text{ep} \pi^3 x^{n+z} \text{Csc}[\pi z]^3}{1+n+z}$

In[19]:= `-Residue[%, {z, 0}]`

Out[19]= $-\frac{1}{(1+n)^3} (-1)^n \text{ep}\, x^n \left(2+\pi^2+2 n \pi^2+n^2 \pi^2-2 \text{Log}[x]-2 n \text{Log}[x]+\text{Log}[x]^2+2 n \text{Log}[x]^2+n^2 \text{Log}[x]^2\right)$

In[20]:= `Apart[% /. n → n - 1, n]`

Out[20]= $\frac{2 (-1)^n \text{ep}\, x^{-1+n}}{n^3}-\frac{2 (-1)^n \text{ep}\, x^{-1+n} \text{Log}[x]}{n^2}+\frac{(-1)^n \text{ep}\, x^{-1+n} \left(\pi^2+\text{Log}[x]^2\right)}{n}$

In[21]:= `res2 = Sum[%, {n, 1, Infinity}]`

Out[21]= $\frac{1}{x}\left(-\text{ep}\, \pi^2 \text{Log}[1+x]-\text{ep}\, \text{Log}[x]^2 \text{Log}[1+x]-2 \text{ep}\, \text{Log}[x] \text{PolyLog}[2,-x]+2 \text{ep}\, \text{PolyLog}[3,-x]\right)$

`(*  Numerical check   *)`

In[22]:= `% / ep /. {x → 0.76, ep → 0.3}`

Out[22]= -9.70976

In[23]:= `NIntegrate[int1 / ep /. {x → 0.76, ep → 0.3, z → -0.5 + I * y1}, {y1, -Infinity, Infinity}] / 2 / Pi`

Out[23]= $-9.70976+2.67167\times 10^{-13}\, i$

In[24]:= `res1 + res2`

Out[24]= $\frac{4}{\text{ep}^2 x}-\frac{4 \pi^2}{3 x}-\frac{2 \text{Log}[x]}{\text{ep}\, x}+\frac{7 \text{ep}\, \pi^2 \text{Log}[x]}{6 x}+\frac{\text{ep}\, \text{Log}[x]^3}{3 x}+\frac{17 \text{ep}\, \text{PolyGamma}[2,1]}{3 x}+\frac{1}{x}\left(-\text{ep}\, \pi^2 \text{Log}[1+x]-\text{ep}\, \text{Log}[x]^2 \text{Log}[1+x]-2 \text{ep}\, \text{Log}[x] \text{PolyLog}[2,-x]+2 \text{ep}\, \text{PolyLog}[3,-x]\right)$

`(*  This is our result (up to I Pi^(d/2) ) *)`

In[25]:= `(% /. x → T / S) S^(-2-ep)`

Out[25]= $S^{-2-\text{ep}} \left(\frac{4 S}{\text{ep}^2 T}-\frac{4 \pi^2 S}{3 T}-\frac{2 S \text{Log}\left[\frac{T}{S}\right]}{\text{ep}\, T}+\frac{7 \text{ep}\, \pi^2 S \text{Log}\left[\frac{T}{S}\right]}{6 T}+\frac{\text{ep}\, S \text{Log}\left[\frac{T}{S}\right]^3}{3 T}+\frac{17 \text{ep}\, S\, \text{PolyGamma}[2,1]}{3 T}+\frac{1}{T} S \left(-\text{ep}\, \pi^2 \text{Log}\left[1+\frac{T}{S}\right]-\text{ep}\, \text{Log}\left[\frac{T}{S}\right]^2 \text{Log}\left[1+\frac{T}{S}\right]-2 \text{ep}\, \text{Log}\left[\frac{T}{S}\right] \text{PolyLog}\left[2,-\frac{T}{S}\right]+2 \text{ep}\, \text{PolyLog}\left[3,-\frac{T}{S}\right]\right)\right)$