```
<< MB/MB.m
```

```
MB 1.1

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more info in hep-ph/0511200

last modified 06 Mar 08
```

```
(*      Example  4a      *)

F = Gamma[3 / 2 + ep + z] Gamma[-1 - 2 ep - z]
  Gamma[4 ep + z] Gamma[-z] Gamma[1 / 2 - ep - z] / Gamma[1 - 2 ep - z]
```

$$\frac{1}{\Gamma[1-2\,\text{ep}-z]}\,\Gamma[-1-2\,\text{ep}-z]\,\Gamma\left[\frac{1}{2}-\text{ep}-z\right]\,\Gamma[-z]\,\Gamma\left[\frac{3}{2}+\text{ep}+z\right]\,\Gamma[4\,\text{ep}+z]$$

```
(*    Strategy #2    *)

Frules = MBoptimizedRules[F, ep → 0, {}, {ep}]

MBrules::norules : no rules could be found to regulate this integral

{}

(*    Strategy #1    *)

(*   The two residues   *)

-Residue[F, {z, -1 - 2 ep}] + Residue[F, {z, -4 ep}]
```

$$\Gamma\left[\frac{1}{2}-\text{ep}\right]\,\Gamma\left[\frac{3}{2}+\text{ep}\right]\,\Gamma[-1+2\,\text{ep}]\,\Gamma[1+2\,\text{ep}]+\frac{\Gamma\left[\frac{3}{2}-3\,\text{ep}\right]\,\Gamma[4\,\text{ep}]\,\Gamma[-1+2\,\text{ep}]\,\Gamma\left[\frac{1}{2}+3\,\text{ep}\right]}{\Gamma[1+2\,\text{ep}]}$$

```
(*   plus an integral with  the first poles of
 Gamma[4 ep+z]and Gamma[-1-2 ep-z] of the opposite nature     *)


(*  Strategy #2: introduce an auxiliary analytic regularization   *)

F = Gamma[3 / 2 + ep + z] Gamma[-1 - 2 ep - z + y]
  Gamma[4 ep + z] Gamma[-z] Gamma[1 / 2 - ep - z] / Gamma[1 - 2 ep - z]
```

$$\frac{1}{\Gamma[1-2\,\text{ep}-z]}\,\Gamma\left[\frac{1}{2}-\text{ep}-z\right]\,\Gamma[-1-2\,\text{ep}+y-z]\,\Gamma[-z]\,\Gamma\left[\frac{3}{2}+\text{ep}+z\right]\,\Gamma[4\,\text{ep}+z]$$

```
Step1rules = MBoptimizedRules[F, y → 0, {}, {ep, y}]

MBrules::norules : no rules could be found to regulate this integral
```

$$\left\{\left\{\text{ep}\to\frac{1}{2},\ y\to\frac{7}{4}\right\},\ \left\{z\to-\frac{5}{4}\right\}\right\}$$

```
con1 = MBcontinue[F, y → 0, Step1rules]
```

Level 1

Taking −residue in z = −1 − 2 ep + y

Level 2

Integral {1}

2 integral(s) found

$$\left\{\left\{\text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2}+\text{ep}-\text{y}\right]\text{Gamma}[1+2\,\text{ep}-\text{y}]\,\text{Gamma}\left[\frac{1}{2}-\text{ep}+\text{y}\right]\text{Gamma}[-1+2\,\text{ep}+\text{y}]}{\text{Gamma}[2-\text{y}]},\ \left\{\left\{\text{ep}\to\frac{1}{2},\ \text{y}\to 0\right\},\ \{\}\right\}\right]\right\},$$

$$\text{MBint}\left[\frac{1}{\text{Gamma}[1-2\,\text{ep}-\text{z}]}\,\text{Gamma}\left[\frac{1}{2}-\text{ep}-\text{z}\right]\text{Gamma}[-1-2\,\text{ep}+\text{y}-\text{z}]\,\text{Gamma}[-\text{z}]\right.$$

$$\left.\text{Gamma}\left[\frac{3}{2}+\text{ep}+\text{z}\right]\text{Gamma}[4\,\text{ep}+\text{z}],\ \left\{\left\{\text{ep}\to\frac{1}{2},\ \text{y}\to 0\right\},\ \left\{\text{z}\to-\frac{5}{4}\right\}\right\}\right]\Big\}$$

**exp1 = MBexpand[con1, 1, {y, 0, 0}]**

$$\left\{\text{MBint}\left[\text{Gamma}\left[\frac{1}{2}-\text{ep}\right]\text{Gamma}\left[\frac{3}{2}+\text{ep}\right]\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}[1+2\,\text{ep}],\ \left\{\left\{\text{ep}\to\frac{1}{2},\ \text{y}\to 0\right\},\ \{\}\right\}\right],\right.$$

$$\text{MBint}\left[\frac{1}{\text{Gamma}[1-2\,\text{ep}-\text{z}]}\,\text{Gamma}[-1-2\,\text{ep}-\text{z}]\,\text{Gamma}\left[\frac{1}{2}-\text{ep}-\text{z}\right]\right.$$

$$\left.\left.\text{Gamma}[-\text{z}]\,\text{Gamma}\left[\frac{3}{2}+\text{ep}+\text{z}\right]\text{Gamma}[4\,\text{ep}+\text{z}],\ \left\{\left\{\text{ep}\to\frac{1}{2},\ \text{y}\to 0\right\},\ \left\{\text{z}\to-\frac{5}{4}\right\}\right\}\right]\right\}$$

**con2 = Table[MBcontinue[exp1[[i, 1]], ep → 0, exp1[[i, 2]]], {i, Length[exp1]}]**

```
Level 1

1 integral(s) found

Level 1

Taking +residue in z = -1 - 2 ep

Taking +residue in z = -4 ep

Taking +residue in z = -1 - 4 ep

Level 2

Integral {1}

Integral {2}

Integral {3}

4 integral(s) found
```

$$\left\{\left\{\text{MBint}\left[\text{Gamma}\left[\frac{1}{2}-\text{ep}\right]\text{Gamma}\left[\frac{3}{2}+\text{ep}\right]\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}[1+2\,\text{ep}],\ \{\{\text{ep}\to 0,\ \text{y}\to 0\},\ \{\}\}\right]\right\},\right.$$

$$\left\{\left\{\text{MBint}\left[-\text{Gamma}\left[\frac{1}{2}-\text{ep}\right]\text{Gamma}\left[\frac{3}{2}+\text{ep}\right]\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}[1+2\,\text{ep}],\ \{\{\text{ep}\to 0,\ \text{y}\to 0\},\ \{\}\}\right]\right\},\right.$$

$$\left\{\text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2}-3\,\text{ep}\right]\text{Gamma}[4\,\text{ep}]\,\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}\left[\frac{1}{2}+3\,\text{ep}\right]}{\text{Gamma}[1+2\,\text{ep}]},\ \{\{\text{ep}\to 0,\ \text{y}\to 0\},\ \{\}\}\right]\right\},$$

$$\left\{\text{MBint}\left[-\frac{\text{Gamma}\left[\frac{1}{2}-3\,\text{ep}\right]\text{Gamma}[2\,\text{ep}]\,\text{Gamma}\left[\frac{3}{2}+3\,\text{ep}\right]\text{Gamma}[1+4\,\text{ep}]}{\text{Gamma}[2+2\,\text{ep}]},\ \{\{\text{ep}\to 0,\ \text{y}\to 0\},\ \{\}\}\right]\right\},$$

$$\text{MBint}\left[\frac{1}{\text{Gamma}[1-2\,\text{ep}-\text{z}]}\text{Gamma}[-1-2\,\text{ep}-\text{z}]\,\text{Gamma}\left[\frac{1}{2}-\text{ep}-\text{z}\right]\right.$$

$$\left.\left.\left.\text{Gamma}[-\text{z}]\,\text{Gamma}\left[\frac{3}{2}+\text{ep}+\text{z}\right]\text{Gamma}[4\,\text{ep}+\text{z}],\ \left\{\{\text{ep}\to 0,\ \text{y}\to 0\},\ \left\{\text{z}\to -\frac{5}{4}\right\}\right\}\right]\right\}\right\}$$

```
MBmerge[%]
```

$$\left\{\text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2}-3\,\text{ep}\right]\text{Gamma}[4\,\text{ep}]\,\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}\left[\frac{1}{2}+3\,\text{ep}\right]}{\text{Gamma}[1+2\,\text{ep}]}-\right.\right.$$

$$\left.\frac{\text{Gamma}\left[\frac{1}{2}-3\,\text{ep}\right]\text{Gamma}[2\,\text{ep}]\,\text{Gamma}\left[\frac{3}{2}+3\,\text{ep}\right]\text{Gamma}[1+4\,\text{ep}]}{\text{Gamma}[2+2\,\text{ep}]},\ \{\{\text{ep}\to 0,\ \text{y}\to 0\},\ \{\}\}\right],$$

$$\text{MBint}\left[\frac{1}{\text{Gamma}[1-2\,\text{ep}-\text{z}]}\text{Gamma}[-1-2\,\text{ep}-\text{z}]\,\text{Gamma}\left[\frac{1}{2}-\text{ep}-\text{z}\right]\text{Gamma}[-\text{z}]\right.$$

$$\left.\left.\text{Gamma}\left[\frac{3}{2}+\text{ep}+\text{z}\right]\text{Gamma}[4\,\text{ep}+\text{z}],\ \left\{\{\text{ep}\to 0,\ \text{y}\to 0\},\ \left\{\text{z}\to -\frac{5}{4}\right\}\right\}\right]\right\}$$

```
exp2 = MBexpand[%, Exp[2 ep EulerGamma], {ep, 0, 0}]
```

```
MBmerge[%]
```

$$\left\{\text{MBint}\left[-\frac{1}{96\,\text{ep}^2}\pi\left(6-6\,\text{ep}\left(-6+2\,\text{EulerGamma}-3\,\text{PolyGamma}\left[0,\frac{1}{2}\right]+3\,\text{PolyGamma}\left[0,\frac{3}{2}\right]\right)+\right.\right.\right.$$
$$\text{ep}^2\left(12\,\text{EulerGamma}^2+35\,\pi^2-36\,\text{EulerGamma}\left(2+\text{PolyGamma}\left[0,\frac{1}{2}\right]-\text{PolyGamma}\left[0,\frac{3}{2}\right]\right)+\right.$$
$$3\left(-44+9\,\text{PolyGamma}\left[0,\frac{1}{2}\right]^2+12\,\text{PolyGamma}\left[0,\frac{3}{2}\right]+9\,\text{PolyGamma}\left[0,\frac{3}{2}\right]^2-\right.$$
$$\left.\left.\left.\left.\left.6\,\text{PolyGamma}\left[0,\frac{1}{2}\right]\left(2+3\,\text{PolyGamma}\left[0,\frac{3}{2}\right]\right)\right)\right)\right)\right),\ \{\{\text{ep}\to 0,\ \text{y}\to 0\},\ \{\}\}\right],$$
$$\text{MBint}\left[\frac{\text{Gamma}[-1-z]\,\text{Gamma}\left[\frac{1}{2}-z\right]\,\text{Gamma}[-z]\,\text{Gamma}[z]\,\text{Gamma}\left[\frac{3}{2}+z\right]}{\text{Gamma}[1-z]},\right.$$
$$\left\{\{\text{ep}\to 0,\ \text{y}\to 0\},\right.$$
$$\left.\left.\left\{z\to -\frac{5}{4}\right\}\right\}\right]\Bigg\}$$

```
(*      Example  4b      *)
```

In[26]:=
```
F = Gamma[-1 / 2 + ep + z] Gamma[1 + ep + z] Gamma[3 / 2 - ep - z] Gamma[-z]
```

Out[26]=
$$\text{Gamma}\left[\frac{3}{2}-\text{ep}-z\right]\text{Gamma}[-z]\,\text{Gamma}\left[-\frac{1}{2}+\text{ep}+z\right]\text{Gamma}[1+\text{ep}+z]$$

```
(*    Strategy #1:
 there are no poles. Expand the integrand in epsilon. However,
the contour cannot be a straight line.   *)

(*    Strategy #2        *)
```

In[27]:=
```
F = Gamma[-1 / 2 + ep + z] Gamma[1 + ep + z] Gamma[3 / 2 - ep - z] Gamma[-z]
```

Out[27]=
$$\text{Gamma}\left[\frac{3}{2}-\text{ep}-z\right]\text{Gamma}[-z]\,\text{Gamma}\left[-\frac{1}{2}+\text{ep}+z\right]\text{Gamma}[1+\text{ep}+z]$$

In[28]:=
```
Frules = MBoptimizedRules[F, ep → 0, {}, {ep}]
```

MBresidues::contour : contour starts and/or ends on a pole of Gamma[1 + ep + z]

MBresidues::contour : contour starts and/or ends on a pole of Gamma[1 + ep + z]

MBresidues::contour : contour starts and/or ends on a pole of Gamma[1 + ep + z]

General::stop : Further output of MBresidues::contour will be suppressed during this calculation. ≫

Out[28]=
```
$Aborted
```

```
(*The integral of Gamma[a+s] Gamma[b+s] Gamma[c-s] Gamma[dd-s]*)
```

In[29]:=
```
Mel40[a_, b_, c_, d_] := Gamma[a + c] Gamma[a + d] Gamma[b + c] Gamma[b + d] / Gamma[a + b + c + d];
```

In[30]:=
```
Mel40[-1 / 2 + ep, 1 + ep, 3 / 2 - ep, 0]
```

Out[30]=
$$\frac{3\sqrt{\pi}\,\text{Gamma}\left[-\frac{1}{2}+\text{ep}\right]\text{Gamma}[1+\text{ep}]}{4\,\text{Gamma}[2+\text{ep}]}$$