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CONSTRUCTING SOLUTIONS FOR THE GENERALIZED HÉNON–HEILES SYSTEM THROUGH THE PAINLEVÉ TEST

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The generalized Hénon–Heiles system is considered. New special solutions for two nonintegrable cases are obtained using the Painlevé test. The solutions have the form of the Laurent series depending on three parameters. One parameter determines the singularity-point location, and the other two parameters determine the coefficients in the Laurent series. For certain values of these two parameters, the series becomes the Laurent series for the known exact solutions. It is established that such solutions do not exist in other nonintegrable cases.

Keywords: nonintegrable systems, Painlevé test, singularity analysis, polynomial potential, Hénon–Heiles system, Laurent series, elliptic functions

1. Painlevé property and integrability

A Hamiltonian system defined in a $2s$-dimensional phase space is said to be *fully integrable*, or Liouville integrable, if there exist s independent commuting integrals of motion. In this case, the equations of motion are separable (at least, in principle), and the solution can be obtained in quadratures.

In problems in mechanics and field theory, the coordinates and time are assumed to be real. On the other hand, the integrability of the equations of motion depends on the behavior of their solutions as functions of complex time and also complex spatial coordinates in the case of field theory. The idea of interpreting time as a complex variable and requiring the mechanical-problem solutions to be single-valued functions meromorphic in the entire complex plane was first advanced by Kovalevskaya [1]. This idea led Kovalevskaya to a remarkable result [1] (also see [2], [3]): a new integrable case for the motion of a massive solid body around a fixed point was discovered. This case is currently known as the Kovalevskaya case. This result demonstrated that the analytic theory of differential equations can be fruitfully applied to physical problems. An important step in the development of this theory was the Painlevé classification of ordinary differential equations (ODEs) with respect to the types of singularities in their solutions.

We formulate the Painlevé property for ODEs. We interpret the solution of a system of ODEs as an analytic function that can have isolated singularities [4], [5]. A singularity point is called a *critical point* if the function changes its value when the singularity point is traced around. Otherwise, the singularity point is said to be *noncritical*. A singularity point whose location depends on the initial conditions is called a *movable singularity*.[1]

Definition [6]**.** A system of ODEs has the *Painlevé property* if its general solution has no movable critical singularities.

In the neighborhood of the singularity at the point t_0, an arbitrary solution of such a system can be expanded in a Laurent series containing a finite number of terms with negative powers of the difference

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[1]For systems where the Hamiltonian is independent of time, all singularity points are movable.

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$t - t_0$. In a number of cases, a system that does not have the Painlevé property can be reduced to the form of a system that does have it by changing the variables. In such cases, we say that the initial system has the *weak Painlevé property*.

It has been shown that many mechanical systems can be completely integrated only if the values of the parameters are such that the system has the Painlevé property or the weak Painlevé property [7]–[9]. Some arguments elucidating the connection between the Painlevé analysis and the existence of the integrals of motion were given in [10]. It has been proved that if a system has complex and irrational "resonances," then it is not algebraically integrable [11] (also see [12] and the literature therein). But the integrability of an arbitrary system that has the Painlevé property has not yet been proved, and an algorithm for constructing the additional integral from the Painlevé analysis has not been constructed. An example of a system that is integrable in quadratures but does not have the Painlevé property can be readily given [13]: $H = p^2/2 + f(x)$, where $f(x)$ is a polynomial of fifth or higher order. This system can be trivially integrated in quadratures, and its general solution is not a meromorphic function.

The term Painlevé test refers to any algorithm aimed at checking if the necessary conditions for the Painlevé property of a differential equation are satisfied. The algorithm constructed and used by Painlevé himself to identify all second-order ODEs that have the Painlevé property [6] is known as the α-method. The Kovalevskaya method [1] is less general but much simpler than the α-method.

Developing the Kovalevskaya method further, Ablowitz, Ramani, and Segur [14] constructed a new algorithm for the Painlevé test for ODEs. They were the first to notice the connection between nonlinear partial differential equations integrable by the inverse scattering method and equations that have the Painlevé property. Later, the Painlevé property for partial differential equations was formulated, and the corresponding Painlevé test (the WTC method) was constructed [15], [16] (also see [17]–[20]).

Based on the Painlevé test, an algorithm for obtaining special solutions of an ODE in the form of finite expansions with respect to the unknown function $\varphi(t - t_0)$ was constructed [21]. The function $\varphi(t - t_0)$ and the coefficients in the expansion are solutions of a certain system of ODEs. The latter system is often simpler than the initial equation. This method was used to obtain exact special solutions of nonintegrable ODEs [22]. The four-parameter generalization of the exact three-parameter solution of the ninth Bianchi cosmology model, i.e., the Mixmaster, was obtained in [23] using a Painlevé test based on the perturbation theory [18].

The objective in this article is to obtain new special solutions of the generalized Hénon–Heiles system using a Painlevé test. In contrast with [22], we express the solutions as formal Laurent series (possibly multiplied by $\sqrt{t - t_0}$). We then obtain the convergency domain for these series.

2. The Hénon–Heiles Hamiltonian

In the 1960s, models of the motion of stars in a cylindrically symmetric, time-independent potential were intensively investigated by astronomers [24], [25]. Because the potential is symmetric, the three-dimensional problem reduces to the two-dimensional one. But obtaining the analytic form for the second integral of the resulting system—for example, in the form of a polynomial with respect to the phase variables—is an insolvable problem, even for relatively simple polynomial potentials. To learn whether the unknown integral exists, Hénon and Heiles investigated the behavior of the trajectories, integrating the equations of motion numerically [25]. Emphasizing that their choice of the potential was not based on experimental data, they proposed the Hamiltonian

$$H = \frac{1}{2}(x_t^2 + y_t^2 + x^2 + y^2) + x^2 y - \frac{1}{3}y^3.$$

On one hand, this Hamiltonian is sufficiently simple that the trajectories can be obtained easily, and, on the other hand, sufficiently complex that the obtained trajectories are by no means trivial. Indeed, for small energies the Hénon–Heiles system seems integrable: the trajectories obtained from the numerical integration lie on two-dimensional surfaces for any initial conditions, i.e., the situation is same as in the case where the second independent integral does exist. At the same time, many of these surfaces disintegrate as the energy increases, which points toward the absence of the second integral. Subsequent numerical studies [26], [27] demonstrated that in the complex t plane, the singularity points for the solutions of the equations of motion are grouped into self-similar spirals. The resulting extremely complex singularity distributions form the boundary beyond which the solution cannot be analytically continued.

The generalized Hénon–Heiles system is described by the Hamiltonian

$$H = \frac{1}{2}(x_t^2 + y_t^2 + \lambda x^2 + y^2) + x^2 y - \frac{C}{3} y^3, \tag{1}$$

and the corresponding system of the equations of motion are

$$\begin{aligned} x_{tt} &= -\lambda x - 2xy, \\ y_{tt} &= -y - x^2 + Cy^2, \end{aligned} \tag{2}$$

where $x_{tt} \equiv d^2x/dt^2$, $y_{tt} \equiv d^2y/dt^2$, and λ and C are numerical parameters.

The Painlevé analysis yields the following integrable cases (2):

a. $C = -1$, $\lambda = 1$;

b. $C = -6$, λ is an arbitrary number;

c. $C = -16$, $\lambda = 1/16$.

The Hénon–Heiles system is a model that not only is intensively investigated by different mathematical methods[2] but also is widely used in physics, in particular, in gravitation theory [29]–[31] and plasma theory [32]. Models that result from adding nonpolynomial terms in Hamiltonian (1) are also intensively investigated [33]–[35].

3. Nonintegrable cases

General solutions of the Hénon–Heiles system are known only for integrable cases [35]. In other cases, not only exact four-parameter solutions but also exact three-parameter solutions have not yet been obtained.

As a system of two second-order ODEs, the Hénon–Heiles system is equivalent to a fourth-order equation[3]

$$y_{tttt} = (2C - 8) y_{tt} y - (4\lambda + 1) y_{tt} + 2(C+1) y_t^2 + \frac{20C}{3} y^3 + (4C\lambda - 6) y^2 - 4\lambda y - 4H, \tag{3}$$

where H is the energy of the system. Special solutions of these equations can be obtained by assuming that y is a solution of a simpler ODE. A well-known example is the two-parameter solutions in the form of elliptic Weierstrass functions [36] satisfying the first-order equation

$$y_t^2 = \tilde{\mathcal{A}} y^3 + \widetilde{\mathcal{B}} y^2 + \widetilde{\mathcal{C}} y + \widetilde{\mathcal{D}}, \tag{4}$$

[2] The history of studies of the generalized Hénon–Heiles system is described in [28].

[3] If $y(t)$ is known, the function $x^2(t)$ can be obtained as a solution of a linear equation. System (2) is invariant under replacing x with $-x$.

where $\tilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$, $\widetilde{\mathcal{C}}$, and $\widetilde{\mathcal{D}}$ are constants.

Timoshkova [37] generalized Eq. (4) to

$$y_t^2 = \tilde{\mathcal{A}}y^3 + \widetilde{\mathcal{B}}y^2 + \widetilde{\mathcal{C}}y + \widetilde{\mathcal{D}} + \widetilde{\mathcal{G}}y^{5/2} + \widetilde{\mathcal{E}}y^{3/2} \qquad (5)$$

and obtained new one-parameter sets of solutions for the Hénon–Heiles system in nonintegrable cases ($C = -4/3$ or $C = -16/5$, λ is an arbitrary number). Four equations with $\widetilde{\mathcal{G}} \neq 0$ or $\widetilde{\mathcal{E}} \neq 0$ correspond to each pair of C and λ values. It is noteworthy that such equations can be written for $\widetilde{\mathcal{D}} = 0$; therefore, substituting $y = \varrho^2$ leads to the equation

$$\varrho_t^2 = \frac{1}{4}(\tilde{\mathcal{A}}\varrho^4 + \widetilde{\mathcal{G}}\varrho^3 + \widetilde{\mathcal{B}}\varrho^2 + \widetilde{\mathcal{E}}\varrho + \widetilde{\mathcal{C}}). \qquad (6)$$

The general solution of this equation depends on one arbitrary parameter and can be expressed in either elementary or elliptic functions.

4. Painlevé test results for the Hénon–Heiles system

The Ablowitz–Ramani–Segur algorithm of the Painlevé test is extremely useful for obtaining the solutions as formal Laurent series. Let the behavior of the solution in the neighborhood of the singularity point t_0 be algebraic, i.e., the solutions tend to infinity as

$$x = a_\alpha(t - t_0)^\alpha, \qquad y = b_\beta(t - t_0)^\beta, \qquad (7)$$

where α, β, a_α, and b_β are some constants. Of course, the real parts of α and β should be negative, and $a_\alpha \neq 0$, and $b_\beta \neq 0$.

If α and β are integer, substituting expansions of the form

$$x = a_\alpha(t - t_0)^\alpha + \sum_{k=1}^{N_{\max}} a_{k+\alpha}(t - t_0)^{k+\alpha}, \qquad y = b_\beta(t - t_0)^\beta + \sum_{k=1}^{N_{\max}} b_{k+\beta}(t - t_0)^{k+\beta}$$

allows reducing the system of differential equations to a set of sequentially solved linear algebraic systems with respect to the coefficients a_k and b_k. In the general case, the exact solutions in the form of formal Laurent series can be obtained only if an infinite number of the systems are solved ($N_{\max} = \infty$). On the other hand, it is possible to obtain solutions to the accuracy of $\mathcal{O}(t^{N_{\max}})$ by solving a finite number of systems. The linear algebraic systems can be sequentially solved by computer using computer algebra software, such as REDUCE [38], [39] or Mathematica [40]. But for such a computer solution, it is necessary to predetermine the values of the constants α, β, a_α, and b_β and the numbers of the systems whose determinants are equal to zero. The coefficients of the powers of t corresponding to such systems may contain new arbitrary parameters. These powers are often called the resonances. The computer solutions can be obtained only after such systems are investigated. All this necessary information can be obtained through a Painlevé test (see, e.g., [9]). Moreover, the Painlevé analysis results help to identify cases where it is useful to include terms with fractional powers of $t - t_0$ in the expansions.

There exist two different behavior types for the solutions of system (2) in the neighborhood of the singularity [9], [27], [41] (see Table 1). The values of the variable r indicate the resonances: $r = -1$

corresponds to t_0, and $r = 0$ in case 2 corresponds to the situation where the leading term is proportional to the arbitrary parameter c_1. The remaining values of r determine the powers of t where new arbitrary coefficients appear as solutions of the linear systems with zero determinant, namely, $t^{\alpha+r}$ for x and $t^{\beta+r}$ for y. In one of the cases, the existence of negative resonances other than the resonance $r = -1$ that always exists means that this case corresponds to a singular solution and not to the general solution.

Table 1

Case 1	Case 2 $\big(\beta < \operatorname{Re}\alpha < 0\big)$
$\alpha = -2$	$\alpha = \big(1 \pm \sqrt{1 - 48/C}\,\big)/2$
$\beta = -2$	$\beta = -2$
$a_\alpha = \pm 3\sqrt{2 + C}$	$a_\alpha = c_1$ (arbitrary)
$b_\beta = -3$	$b_\beta = 6/C$
$r = -1,\ 6,\ 5/2 - \big(\sqrt{1 - 24(1 + C)}\,\big)/2,$ $5/2 + \big(\sqrt{1 - 24(1 + C)}\,\big)/2$	$r = -1,\ 0,\ 6,\ \mp\sqrt{1 - 48/C}$

It is necessary for the integrability of system (2) that all values of α and r be integer or rational and that all systems with zero determinants have solutions for any values of the free parameters entering these systems. This is possible only in the integrable cases *a*–*c* (see Sec. 2).

In the search for special solutions, it is interesting to consider the values of C for which α and r are integer or rational numbers either only in case 1 or only in case 2. We identify all cases where there exists a special (not singular) solution that can be expressed in the form of a three-parameter Laurent series (possibly multiplied by $\sqrt{t - t_0}\,$). Because the values of r must be natural, the general solution can be expressed in the form of a Laurent series for either $C = -1$ and $C = -4/3$ (case 1) or $C = -16/5$, $C = -6$, and $C = -16$ (case 2, $\alpha = (1 - \sqrt{1 - 48/C}\,)/2$) and also $C = -2$, in which case the two behavior types for the solutions in the singularity neighborhood merge into one. We consider all these alternatives.

For $C = -2$, we obtain the inconsistency $a_\alpha = 0$ (see case 1) because our assumption is not satisfied and the behavior of the solution in the singularity neighborhood is not algebraic: the leading order contains logarithmic terms [9]. For $C = -6$ and any value of λ, the exact four-parameter solutions are known. In the cases where $C = -1$ and $C = -16$, substituting the unknown functions in the form of the Laurent series leads to the respective equations $\lambda = 1$ and $\lambda = 1/16$ for λ. Therefore, the solutions that are free of logarithms exist only in the integrable cases. Hence, special solutions in the form of the Laurent series depending on three parameters can exist only in two nonintegrable cases, namely, for $C = -16/5$ and $C = -4/3$. It is remarkable that these Laurent series generalize the exact solutions obtained in [37].

5. New solutions

We consider the Hénon–Heiles system for $C = -16/5$. In case 2, we obtain $\alpha = -3/2$ and $r = -1, 0, 4, 6$; therefore, x should be sought in a form such that the expansion of x^2 in the Laurent series in the neighborhood of t_0 begins with $(t - t_0)^{-3}$. Let $t_0 = 0$. Substituting

$$x = \sqrt{t}\bigg(c_1 t^{-2} + \sum_{k=-1}^{\infty} a_k t^k\bigg), \qquad y = -\frac{15}{8}t^{-2} + \sum_{k=-1}^{\infty} b_k t^k$$

in system (2), we obtain the sequence of linear systems for the coefficients a_k and b_k:

$$
\begin{aligned}
&(k^2-4)a_k + 2c_1 b_k = -\lambda a_{k-2} - 2\sum_{j=-1}^{k-1} a_j b_{k-j-2}, \\
&\big((k-1)k-12\big)b_k = -b_{k-2} - \sum_{j=-2}^{k-1} a_j a_{k-j-3} - \frac{16}{5}\sum_{j=-1}^{k-1} b_j b_{k-j-2}.
\end{aligned}
\tag{8}
$$

The determinants of the systems corresponding to the values $k = 2$ and $k = 4$ are equal to zero. To determine a_2 and b_2, we obtain the system

$$
\begin{aligned}
&c_1\big(557056c_1^8 + (15552000\lambda - 4860000)c_1^4 + 864000000b_2 + \\
&\qquad + 108000000\lambda^2 - 67500000\lambda + 10546875\big) = 0, \\
&818176c_1^8 + (15660000\lambda - 4893750)c_1^4 - 810000000b_2 - 6328125 = 0.
\end{aligned}
\tag{9}
$$

It is easy to see that this system contains no terms proportional to a_2. Therefore, a_2 is the new integration constant. Disregarding the solution with $c_1 = 0$, we obtain a system for $\tilde{c}_1 \equiv c_1^4$ and b_2. This system has the solutions

$$
\tilde{c}_1 = \frac{1125\big(4\sqrt{35(2048\lambda^2 - 1280\lambda + 387)} - 1680\lambda + 525\big)}{167552},
$$

$$
b_2 = -\frac{(10944\lambda - 3420)\sqrt{35(2048\lambda^2 - 1280\lambda + 387)} - 4403456\lambda^2 + 2752160\lambda - 789065}{117956608}
$$

and

$$
\tilde{c}_1 = \frac{1125\big(-4\sqrt{35(2048\lambda^2 - 1280\lambda + 387)} - 1680\lambda + 525\big)}{167552},
$$

$$
b_2 = \frac{(10944\lambda - 3420)\sqrt{35(2048\lambda^2 - 1280\lambda + 387)} - 4403456\lambda^2 + 2752160\lambda - 789065}{117956608}.
$$

We thus obtain the new integration constant a_2. But we needed to fix c_1, and the number of arbitrary parameters remains equal to two. It is easy to verify that for $k = 4$, the system reduces to one equation and b_4 is the new arbitrary parameter. We hence obtain a formal solution depending on three parameters, namely, t_0, a_2, and b_4.

We can now obtain the solution to any required accuracy using computer algebra software. For a given value of λ, we choose one of the possible values of c_1, and then a_j and b_j are obtained automatically. We thus obtain four three-parameter solutions for $C = -16/5$, corresponding to the four exact one-parameter solutions obtained in [37]. The Laurent series for the exact solutions coincide with the corresponding two-parameter series for certain values of these parameters. The case where $\lambda = 1/9$ is considered in the appendix as an example.

For $C = -4/3$, the situation is analogous to that in the case examined above. In case 1, we have $r = -1, 1, 4, 6$. Substituting

$$
x = \sqrt{6}\,t^{-2} + \sum_{k=-1}^{\infty} d_k t^k, \qquad y = -3t^{-2} + \sum_{k=-1}^{\infty} f_k t^k
$$

in system (2), we obtain the sequence of linear systems for the coefficients d_k and f_k:

$$
\begin{aligned}
&\big((k-1)k-6\big)d_k + 2\sqrt{6}\,f_k = -\lambda d_{k-2} - 2\sum_{j=-1}^{k-1} d_j f_{k-j-2},\\
&\big((k-1)k-8\big)f_k + 2\sqrt{6}\,d_k = -f_{k-2} - \sum_{j=-1}^{k-1} d_j d_{k-j-2} - \frac{4}{3}\sum_{j=-1}^{k-1} f_j f_{k-j-2}.
\end{aligned}
\tag{10}
$$

These systems have a zero determinant for $k=-1,2,4$. In the case where $k=-1$, the system always has a solution, and f_{-1} is the new parameter fixed by solving the system with $k=2$. If

$$
f_{-1} = \pm\sqrt{\frac{\sqrt{7(1216\lambda^2-1824\lambda+783)}\,-140\lambda+105}{385}}
$$

or

$$
f_{-1} = \pm\sqrt{\frac{-\sqrt{7(1216\lambda^2-1824\lambda+783)}\,-140\lambda+105}{385}}\,,
$$

then the system has solutions, and f_2 is the new arbitrary parameter. For $k=4$, in analogy with the case where $C=-16/5$, the system reduces to one equation, and f_4 is the new arbitrary parameter. Hence, in the case where $C=-4/3$, we again have four three-parameter (t_0, f_2, f_4) solutions corresponding to the four exact one-parameter solutions obtained in [37]. The Laurent series for the exact solutions coincide with the corresponding two-parameter series for certain values of these parameters.

Once a formal series is obtained, the question of its convergence naturally arises. It was proved in [41], [42] that the solutions of the generalized Hénon–Heiles system obtained as formal psi-series have a nonzero convergency domain. The case where $C=-16/5$ and $\lambda=1/9$ is considered in the appendix as an example. For this case, it is proved that if $|a_2|\le 1$ and $|b_4|\le 1$, then the Laurent series converges in the ring $0<|t-t_0|\le(1-\varepsilon)$, where ε is an arbitrary positive number. The convergence of the series for other values of C and λ can be investigated similarly.

6. Conclusions

The Painlevé analysis allows not only identifying the integrable cases of dynamical systems but also constructing special solutions, even in nonintegrable cases.

Three-parameter solutions in the form of Laurent series are obtained for the Hénon–Heiles system with $C=-16/5$ or $C=-4/3$ and an arbitrary λ. For certain values of the two parameters, these solutions coincide with the known exact solutions. The series have a nonzero convergence domain. It is shown using the Painlevé test that the nonintegrable Hénon–Heiles system has special solutions in the form of a three-parameter Laurent series only for the abovementioned values of C. In these cases, the probability of obtaining the exact three-parameter solutions of Eq. (3) is highest because nothing impedes their existence in the form of single-valued functions.

Appendix

We consider the case where $C=-16/5$ and $\lambda=1/9$. There are two possible forms of Eq. (5). The first possible form is

$$
y_t^2 + \frac{32}{15}y^3 + \frac{4}{9}y^2 \pm \frac{8i}{\sqrt{135}}y^{5/2} = 0,
$$

and depending on the sign of the last term, we obtain either

$$y = -\frac{5}{3\big(1 - 3\sin\big((t - t_0)/3\big)\big)^2} \tag{11a}$$

for the case with the plus sign or

$$y = -\frac{5}{3\big(1 + 3\sin\big((t - t_0)/3\big)\big)^2} \tag{11b}$$

for the case with the minus sign. The other possible form is

$$y_t^2 + \frac{32}{15}y^3 + \frac{1748}{1683}y^2 \pm \frac{8\sqrt{65}}{15\sqrt{561}}y^{5/2} \pm \frac{8125\sqrt{65}}{20196\sqrt{561}}y^{3/2} + \frac{333125}{7553304}y = 0,$$

and the solutions can be obtained in the form of the elliptic Jacobi functions.

The solutions of system (9) are

$$\left\{\tilde{c}_1 = \frac{625}{128}, \quad b_2 = -\frac{1819}{663552}\right\}, \qquad \left\{\tilde{c}_1 = -\frac{8125}{23936}, \quad b_2 = -\frac{8700683}{1364926464}\right\},$$

and we obtain four types of the function y:

$$\begin{aligned} y = &-\frac{15}{8}t^{-2} + \frac{5\sqrt{2}}{32}t^{-1} - \frac{205}{2304} + \frac{115\sqrt{2}}{13824}t - \frac{1819}{663552}t^2 + \\ &+ \left(\frac{741719\sqrt{2}}{1528823808} + \frac{5\sqrt[4]{2}}{12}a_2\right)t^3 + b_4t^4 + \dots, \end{aligned} \tag{12a}$$

$$\begin{aligned} y = &-\frac{15}{8}t^{-2} - \frac{5\sqrt{2}}{32}t^{-1} - \frac{205}{2304} - \frac{115\sqrt{2}}{13824}t - \frac{1819}{663552}t^2 - \\ &- \left(\frac{741719\sqrt{2}}{1528823808} + \frac{5i\sqrt[4]{2}}{12}a_2\right)t^3 + b_4t^4 + \dots, \end{aligned} \tag{12b}$$

$$\begin{aligned} y = &-\frac{15}{8}t^{-2} + \frac{5i\sqrt{4862}}{5984}t^{-1} - \frac{69335}{430848} - \frac{37745i\sqrt{4862}}{483411456}t - \frac{8700683}{1364926464}t^2 - \\ &- \left(\frac{1148020763i\sqrt{13}\,\sqrt{374}}{3332429743915008} - \frac{5\sqrt{2}}{12}a_2\sqrt[4]{-\frac{13}{374}}\right)t^3 + b_4t^4 + \dots, \end{aligned} \tag{12c}$$

$$\begin{aligned} y = &-\frac{15}{8}t^{-2} - \frac{5i\sqrt{4862}}{5984}t^{-1} - \frac{69335}{430848} + \frac{37745i\sqrt{4862}}{483411456}t - \frac{8700683}{1364926464}t^2 - \\ &- \left(\frac{1148020763\sqrt{13}\,\sqrt{374}}{3332429743915008} + \frac{5i\sqrt{2}}{12}a_2\sqrt[4]{-\frac{13}{374}}\right)t^3 + b_4t^4 + \dots. \end{aligned} \tag{12d}$$

It is easy to verify that series (12a) for $a_2 = -21497\sqrt[4]{2}/42467328$ and $b_4 = -858455/12039487488$ is the Laurent series for solution (11a) and that series (12b) for $a_2 = -21497i\sqrt[4]{2}/42467328$ and $b_4 = -858455/12039487488$ is the Laurent series for solution (11b). For certain values of the parameters, series (12c) and (12d) are the Laurent series for the solutions of Eqs. (9).

As known, the convergence domain for a Laurent series is a ring. We formulate the conditions under which series (12a)–(12d) and the corresponding series for the function x converge in the domain $0 < |t| \leq$

$1-\varepsilon$, where ε is an arbitrary positive number. The geometric-progression sum $S = \sum_{n=0}^{\infty} t^n = 1/(1-t)$ is finite for $|t| \leq 1-\varepsilon$; therefore, the series converges in a given ring if there exists N such that $\forall n > N$, $|a_n| \leq M$ and $|b_n| \leq M$, where M is a real number.

Let $|a_n| \leq M$ and $|b_n| \leq M$ for all $-1 < n < k$. We then obtain the relations

$$|a_k| \leq \frac{2M(k+1) + |\lambda| + 2|c_1|}{|k^2-4|} M, \qquad |b_k| \leq \frac{21Mk + 26M + 5}{5|k^2-k-12|} M \tag{13}$$

from Eq. (8). It is easy to see that there exists N such that if $|a_n| \leq M$ and $|b_n| \leq M$ for $-1 \leq n \leq N$, then $|a_n| \leq M$ and $|b_n| \leq M$ for $-1 \leq n < \infty$. For example, in the case where $M = 1$, we have $N = 8$ for any possible value of c_1. It is easy to verify that if $|a_2| \leq 1$ and $|b_4| \leq 1$, then $|a_n| \leq 1$ and $|b_n| \leq 1$ for $-1 \leq n \leq 8$ and therefore for an arbitrary n. Hence, the Laurent series converges in the ring $0 < |t| \leq 1-\varepsilon$.

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