Construction of doubly periodic solutions via the Poincare–Lindstedt method in the case of massless $\varphi^4$ theory

Oleg Khrustalev$^{a,\ast}$, Sergey Vernov$^b$

$^a$ Department of Physics, Institute for Theoretical Problems of Microphysics of Moscow State University, Moscow State University, Vosbity Gory, Moscow 119899, Russia

$^b$ Institute of Nuclear Physics, Moscow State University, Vosbity Gory, Moscow 119899, Russia

Abstract

Doubly periodic (periodic both in time and in space) solutions for the Lagrange–Euler equation of the $(1+1)$-dimensional scalar $\varphi^4$ theory are studied. Provided that the nonlinear term is small, the Poincare–Lindstedt asymptotic method can be used to find asymptotic solutions in the standing wave form. The principal resonance problem, which arises for zero mass, is solved if the leading-order term is taken in the form of Jacobi elliptic function. To obtain this leading-order term the system REDUCE is used. © 2001 IMACS. Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. Periodic solutions of nonlinear equations

At present time periodic solutions of the nonlinear wave equation:

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} - \frac{\partial^2 \psi(x,t)}{\partial t^2} - g(\psi) = 0,$$

(1)

where $g(\psi)$ denotes a continuous function on $\mathbb{R}$ such that $g(0) = 0$, are studied intensively. Coron [1] has proved that $T$-periodic (in time) solutions of the following problem:

$$\begin{cases}
\frac{\partial^2 \psi(x,t)}{\partial x^2} - \frac{\partial^2 \psi(x,t)}{\partial t^2} - g(\psi) = 0, & x, t \in \mathbb{R}, \quad g(0) = 0, \\
\lim_{|x| \to \infty} \psi(x,t) = 0, & x, t \in \mathbb{R}, \\
\psi(x,t + T) = \psi(x,t), & x, t, T \in \mathbb{R},
\end{cases}$$

(2)

$^\ast$ Corresponding author.

E-mail addresses: khruz@sunny.bog.msu.su (O. Khrustalev), svernov@theory.sinp.msu.ru (S. Vernov).

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can exist only when
\[ g'(0) > \left(\frac{2\pi}{T}\right)^2. \]
In particular, if \( g(\phi) = \sin(\phi) \), there are no \( T \)-periodic solutions when \( T < 2\pi \) (for any period \( T > 2\pi \) an explicit solution is known, see e.g. [2]). When \( g(\phi) = -\sin(\phi) \) or \( g(\phi) = \pm \phi^3 \), then \( \phi \equiv 0 \) is the only solution of (2) (for any period \( T \)). To obtain periodic solutions of massless \( \phi^4 \) theory one has to change the boundary conditions.

Generalizing the Rabinowitz theorem [3], Brezis et al. [4] have proved that for each \( T \) which is a rational multiple of \( \pi \), the following problem:
\[
\begin{cases}
\frac{\partial^2 \psi(x,t)}{\partial x^2} - \frac{\partial^2 \psi(x,t)}{\partial t^2} - g(\psi) = 0, & x, t \in \mathbb{R}, \quad g(0) = 0, \\
\psi(0, t) = \psi(\pi, t) = 0, & t \in \mathbb{R}, \\
\psi(x, t + T) = \psi(x, t), & x, t, T \in \mathbb{R},
\end{cases}
\]
with continuous nondecreasing functions \( g(\psi) \) such that
\[
\lim_{|\psi| \to \infty} \frac{g(\psi)}{|\psi|} = \infty,
\]
and \( 3\alpha, \beta > 0 \) such that
\[
\forall \psi \in \mathbb{R} : \frac{1}{2} \psi \cdot g(\psi) - \int_0^\pi g(\psi) d\psi \geq \beta |g(\psi)| - \alpha
\]
has a nontrivial weak \( T \)-periodic solution.

By a weak solution we mean such function \( \psi \) that
\[
\int_0^\pi \int_0^{\pi} \left[ \psi(v_0 - v_i) + g(\psi) \right] d\psi = 0
\]
for all \( v \in C^2([0, \pi] \times \mathbb{R}) \) satisfying the boundary and periodicity conditions. The proof in [4] is simpler than the original proof of Rabinowitz. In the case \( g(\psi) = |\psi|^n \cdot \psi \), where \( n > 0 \), this theorem has been proved also in [5].

1.2. Two kinds of doubly periodic solutions

Our investigation is dedicated to the construction of doubly periodic classical fields in the \((1 + 1)\)-dimensional \( \phi^4 \) theory. We study the model of an isolated real scalar field \( \psi(x, t) \), described by the Lagrangian density:
\[
\mathcal{L}(\psi) = \frac{1}{2} \left( \psi_x(x, t) - \psi_{xx}(x, t) - M^2 \psi^2(x, t) - \frac{\epsilon}{2} \psi^4(x, t) \right).
\]
The associated Lagrange–Euler equation is:
\[
\frac{\partial^2 \psi(x, t)}{\partial x^2} - \frac{\partial^2 \psi(x, t)}{\partial t^2} - M^2 \psi^2(x, t) - \epsilon \psi^3(x, t) = 0. \quad (3)
\]
There are two classes of doubly periodic solutions for this equation. If we seek fields in the traveling wave form:

$$\phi(x, t) = \phi(x - vt),$$

where $v$ is the velocity of the wave motion, then Eq. (3) reduces to the Duffing's equation [6]. As is known [7], periodic solutions for this equation are Jacob's elliptic functions. Such solutions are well known for both $(1+1)$-dimensional [8] and $(3+1)$-dimensional [9] variants of $\phi^4$ theory. They are used in the Yang–Mills theory [10].

The second class of doubly periodic solutions consists of functions in the standing wave form:

$$\phi(x, t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} C_{nj} \sin(n(x - x_0)) \sin(j\omega(t - t_0)), \quad (4)$$

where $x_0$ and $t_0$ are constants determined by boundary and initial conditions. Eq. (3) is a translation-invariant, so we can restrict our consideration to the case of zero $x_0$ and $t_0$ without loss of generality. We suppose that the function $\phi(x, t)$ is $2\pi$-periodic in space and seek its period in time.

2. Asymptotic solutions in the standing wave form

Let us consider periodic solutions for Eq. (3), provided that $\varepsilon$ is small or, equivalently, that these solutions have small amplitudes. An asymptotic expansion, containing only bounded functions, is called a uniform expansion. The possibility to obtain a uniform expansion, using standard asymptotic methods [12], for example, Poincare–Lindstedt [13] or Krylov–Bogoliubov [14] methods, depends on values of the frequencies in the zero approximation. If $\varepsilon = 0$, then Eq. (3) is linear and has periodic solutions of the form (4) with frequencies in time $\Omega_j = \sqrt{j^2 + M^2}$, where $j \in 1N$. There are two fundamentally different cases.

If the ratio $\Omega_i/\Omega_j$ is irrational for all $i$ and $j$, then it is a non-resonant case. The periodic asymptotic solutions in non-resonant case are well known. To obtain these solutions to any degree of accuracy, standard asymptotic methods can be used.

The resonance case, when there exist two frequencies $\Omega_i$, whose relation is rational, is more difficult. The Krylov–Bogoliubov method and the standard variant of the Poincare–Lindstedt method can be used to find periodic solutions of the form (4) with frequencies in non-resonant case are well known. To obtain these solutions to any degree of accuracy, standard asymptotic methods can be used.

The important example of resonance case is the massless $\phi^4$ theory when $\Omega_j = j$ and all relations of frequencies are rational, which is the principal resonant case. Using standard asymptotic methods, one cannot construct a periodic solution even to the first order in $\varepsilon$. It is possible to transform the differential equations to a system of nonlinear algebraic equations in Fourier coefficients and frequencies using the Poincare–Dulac’s normal form method. The algorithm of this procedure was constructed [15,16] and
realized in the software for symbolic and algebraic computation REDUCE [17–19] (the system NORT [20,21]). But an algorithm to solve the resulting algebraic system has yet to be created.

3. The standing wave solutions for the massless $\phi^4$ theory

3.1. Construction of asymptotic solutions via the Poincare–Lindstedt method

The purpose of this article is the construction of standing wave solutions of Eq. (3) with $M = 0$, using asymptotic methods when $\varepsilon \ll 1$. We use the Poincare–Lindstedt method: introduce the new time $\tilde{t} = \omega t$ and look for a doubly periodic solution of Eq. (3). The $\phi(x, \tilde{t})$ and a frequency (in time) $\omega$ in the form of power series in $\varepsilon$:

$$
\phi(x, \tilde{t}, \varepsilon) \equiv \sum_{n=0}^{\infty} \phi_n(x, \tilde{t}) \varepsilon^n,
$$

$$
\omega(\varepsilon) \equiv 1 + \sum_{n=1}^{\infty} \omega_n \varepsilon^n.
$$

Expanding the Lagrange–Euler equation in a power series in $\varepsilon$ gives a sequence of equations. The leading two equations are:

- to zero order in $\varepsilon$ the equation for $\phi_0$ is:

$$
\frac{\partial^2 \phi_0}{\partial x^2} - \frac{\partial^2 \phi_0}{\partial \tilde{t}^2} = 0; \quad (5)
$$

- to first order in $\varepsilon$ the equation in $\phi_0$, $\phi_1$ and $\omega_1$ is:

$$
\frac{\partial^2 \phi_1}{\partial x^2} - \frac{\partial^2 \phi_1}{\partial \tilde{t}^2} = 2\omega_1 \frac{\partial^2 \phi_0}{\partial \tilde{t}^2} + \phi_3^0(x, \tilde{t}). \quad (6)
$$

Eq. (5) has many periodic solutions. If we select $\phi_0(x, \tilde{t}) = \sin(x) \sin(\tilde{t})$, then the second equation has no periodic solution, because the frequency of the external force $\sin(3x) \sin(3\tilde{t})$ is equal to the frequency of its own oscillations and it is impossible to put this resonance harmonic to zero by selecting only $\omega_1$. In the massless case, the correct selection of not only the frequency $\omega(\varepsilon)$, but also the function $\phi_0(x, \tilde{t})$ allows us to find a uniform expansion.

3.2. The condition of existence of periodic solutions

The general solution, in the standing wave form (4), for Eq. (5) is the function

$$
\phi_0(x, \tilde{t}) = \sum_{n=1}^{\infty} a_n \sin(nx) \sin(n\tilde{t})
$$
with arbitrary \( a_n \). We have to find coefficients \( a_n \) so that the function \( \varphi(x, \tilde{t}) \) is a periodic solution for Eq. (6). If we select \( \varphi(x, \tilde{t}) \) as a double sum:

\[
\varphi(x, \tilde{t}) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} b_{nj} \sin(nx) \sin(j\tilde{t})
\]

with arbitrary \( b_{nj} \), then Eq. (6) can be presented in the form of Fourier series:

\[
\frac{\partial^2 \varphi(x, \tilde{t})}{\partial x^2} - \frac{\partial^2 \varphi(x, \tilde{t})}{\partial \tilde{t}^2} - 2a_n \varphi(x, \tilde{t}) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} R_{nj}(a, b) \sin(nx) \sin(j\tilde{t}) = 0
\]

and is equivalent to the following infinite system of the algebraic equations in Fourier coefficients of the functions \( \varphi(x, \tilde{t}) \) and \( \varphi(x, \tilde{t}) \):

\[
\forall n, j \in \mathbb{N} : R_{nj}(a, b) = 0. \tag{6a}
\]

This system has a subsystem of the equations in the Fourier coefficients of the function \( \varphi_0(x, \tilde{t}) \):

\[
\forall j \in \mathbb{N} : R_{0j}(a) = 0, \tag{7}
\]

where

\[
R_{0j} = 9a_j^2 + 3a_j^2a_{1j} + a_j \left( 6 \sum_{s=1, s \neq j}^{\infty} (2a_s^2 + a_s a_{s+j}) + 3 \sum_{s=1, s \neq j}^{2j-1} a_s a_{s+j} - 32 \omega_1^3 a_j^3 \right) + 3 \sum_{s=1, \sigma(j) = 1}^{\infty} a_s a_{s+j} + 3 \sum_{s=1, \sigma(j) = 1}^{\infty} a_s a_{s+j} + \sum_{p=1}^{\infty} \sum_{s=1, \sigma(j) = 1}^{\infty} a_s a_{s+p}
\]

If the system (7) is solved and all Fourier coefficients of the function \( \varphi_0(x, \tilde{t}) \) are known, then the system (6a) transforms into a system of linear equations: \( R_{0j}(b) = 0 \), which always has only one solution. Hence, we have obtained the necessary and sufficient condition of the existence of periodic solutions for Eq. (6); there exists a periodic function \( \varphi(x, \tilde{t}) \), satisfying Eq. (6), if and only if the Fourier coefficients of the function \( \varphi_0(x, \tilde{t}) \) satisfy system (7). The coefficient \( a_1 \) is a parameter determining the oscillation amplitude. In fact, let \( a_j = c_j a_1 \) and \( \omega_1 = c_1 a_1^2 \), then all polynomials \( R_{0j}(a) \) are proportional to \( a_1^3 \):

\[
R_{0j}(a) = a_1^3 \overline{R}_{0j}(c) \text{ and, therefore, the coefficient } a_1 \text{ can be selected arbitrarily.}
\]

### 3.3. The approximate solution of system (7)

System (7) is very difficult to solve. On the one hand, all \( R_{0j} \) are infinite series and the number of equations is infinite too. On the other hand, each equation of this system is nonlinear. We restrict ourselves to finding a particular solution. To simplify calculations, we assume that the function \( \varphi_0(x, \tilde{t}) \) contains only odd harmonics. Our goal is to find a real solution so we seek \( c_j \in \mathbb{R} \).
To find an approximate solution we apply the Galerkin method: cut off higher diagonal harmonics \( \forall j > N : c_{2j-1} = 0 \) and seek an approximation for \( \varphi_0(x, \tilde{t}) \) in the following form:

\[
\varphi_0(x, \tilde{t}) = a_1 \left\{ \sum_{j=1}^{N} c_{2j-1} \sin((2j - 1)x) \sin((2j - 1)\tilde{t}) \right\}.
\]

Now we have a finite system of nonlinear equations, with the number of equations three times as many as that of the variables. We solve the system of the \( N \) leading equations and substitute the obtained values in the other equations. In principle, any system of \( N \) nonlinear equations in \( N \) unknowns can be solved, using Buchberger algorithm [22–24]. This algorithm has been realized in the standard procedures of the computer algebra system REDUCE: the SOLVE operator and the Groebner basis package. This algorithm allows us to diagonalize this system of nonlinear equations, constructing an equivalent system, which consists of equation in only one variable, equation in two variables and so on. So, we can obtain solution of the system, solving only equations in one variable and substitute the result into the remaining equations.

The operating memory and other parameters of real computers are not enough to solve very difficult systems. For example, using computer with 128 Mbytes operating memory (RAM), we have solved the system, consisting of the \( N \) leading equations of system (7) only for \( N < 6 \). On the other hand, even for these values of \( N \) the solution can be found only with finite accuracy.

The SOLVE operator in REDUCE (in “on rounded” mode) rounds off numbers with accuracy \( \delta = 10^{-11} \). We assume that the system of the \( N \) leading equations is solved if for all \( j \leq N \) the inequalities \( |R_{jj}(c)| < \delta \) are true. We also admit that the value of \( N \) is sufficient to solve system (7) if for all \( j \in \mathbb{N} \) the inequalities \( |R_{jj}(c)| < \delta \) are true.

To find a particular solution of the \( N \) leading equations for any \( N < 50 \) we have written a short program in REDUCE (see Appendix A) and obtained that the minimal sufficient value of \( N \) for \( \delta = 10^{-11} \) is \( N = 8 \) and that the frequency correction is

\[
\omega_1 = 0.28268003454 a_1^2.
\]

We also have found numerical values of the fifteen leading Fourier coefficients of \( \varphi_0(x, \tilde{t}) \). The values of \( c_j \) are very close to the values of the corresponding terms of the following finite sequence:

\[
d_{d_{2j-1}} = \frac{f_{2j-1}}{f_1}, \quad \text{where} \quad f_{2j-1} = \frac{q^{-(1/2)}}{1 + q^{2j-1}}, \quad d_{2j} = 0; \quad j < 23 \quad \text{and} \quad q = 0.0142142623201.
\]

with \( q = 0.0142142623201 \). It is easy to verify that substitution of this finite sequence gives

\[
\forall j \in \mathbb{N} : |R_{jj}(d)| < 10^{-12}.
\]

In other words, the finite sequence \( d \) is an approximate solution of system (7). These numerical calculations help to find the analytical form of \( \varphi_0(x, \tilde{t}) \). The following Table 1 illustrates this interesting result.

3.4. The exact solution of system (7)

The approximate solution of system (7) is found. The aim of this section is to find an exact solution of (7), that is to say to find the function \( \varphi_0(x, \tilde{t}) \) in analytical form. For arbitrary \( q \in (0, 1) \) let us define the
following sequence:

\[
f^\text{def} = \left\{ h \in \mathbb{I}N : f_{2n-1} = \frac{q^{n-1/2}}{1 + q^{2n-1}}, f_{2n} = 0 \right\}.
\]

The terms of the sequence \( f \) are proportional to Fourier coefficients of the Jacobi elliptic function \( cn \) [7]:

\[
\text{cn}(z, k) = \sum_{k=1}^{\infty} f_{2n-1} \cos \left( \frac{(2n-1)z}{4} \right), \quad \text{where} \quad \gamma = \frac{2\pi}{K}, \quad z \in \mathbb{R}.
\]

Let us clarify the notation and point out some properties of the elliptic cosine:

- Basic periods of the doubly periodic function \( \text{cn}(z, k) \) are \( 4K(k) \) and \( 2\text{K}(k) \), where \( K(k) \) is a full elliptic integral, \( \text{K}'(k) = \text{K}(k') \) and \( k' = \sqrt{1-k^2} \).
- The parameter \( q \) in the Fourier expansion can be expressed in terms of elliptic integrals: \( q = e^{-\pi K/K} \).
- The Fourier-series expansion of the function \( \text{cn}(z, k) \) does not include even harmonics. This expansion is valid in the following domain of the complex plane: \( -K' < 2\pi z < K' \), in particular, for \( z \in \mathbb{R} \).
- If \( z \in \mathbb{R} \) and \( k \in (0, 1) \), then \( \text{cn}(z, k) \in \mathbb{R} \).
- The function \( \text{cn}(z, k) \) is a solution of the following differential equation:
\[
\frac{d^2 \text{cn}(z, k)}{dz^2} = (2k^2 - 1) \text{cn}(z, k) - 2k^2 \text{cn}'(z, k). \tag{9}
\]

The latest property means that the infinite sequence of the Fourier coefficients for \( \text{cn}(z, k) \) is a solution of some infinite system of nonlinear algebraic equations. Let us find this system. On the one hand, it is clear from (8) that the Fourier-series expansion for the function \( \text{cn}^3(z, k) \) is:

\[
\text{cn}^3(z, k) = \gamma^3 4^\infty \sum_{j=1}^\infty F(3j) \cos \left( \frac{j \gamma z}{4} \right), \quad \text{where} \quad j = 1, 3, 5, \ldots, +\infty;
\]

\[
F(3j) = 3f_j + 3f_j^2 + f_j \left( 6 \sum_{s=1, s \neq j}^{\infty} \left( f_s f_{2s} + f_s f_{2s+1} + 3 \sum_{r=1, r \neq j}^{2j-1} f_r f_{2r-j} \right) + 3 \sum_{s=1, s \neq j}^{\infty} \sum_{p=1, p \neq 1, p \neq j}^{\infty} f_s f_p f_{1+p-s} + \sum_{s=1, s \neq j}^{\infty} \sum_{p=1, p \neq 2, p \neq j}^{\infty} f_s f_p f_{2+p-s-j} \right)
\]

(in all sums we summarize over only odd numbers). On the other hand, from the differential Eq. (9) it follows that \( F(3j) \) is proportional to \( f_j \), with coefficients of proportionality depending on \( j \):

\[
\forall j: F(3j) = \left( \frac{2(2k^2 - 1)}{\gamma^2} + \frac{j^2}{8} \right) f_j. \tag{10}
\]

Thus, the sequence \( f \) is a nonzero solution of system (10) at all \( q \in (0, 1) \). The following lemma proves the existence of a preferred value of \( q \).

**Lemma.** There exists such value of parameter \( q \in (0, 1) \) that the sequence \( f \) is a real solution of system (7), in addition a value of \( \omega_1 \) also is real.

**Proof.** Inserting the sequence \( f \) into system (7): \( a_j = f_j \) and using system (10), we obtain:

\[
\forall j: R_j(f) = F(3j)(f) + f_j \left( 6 \sum_{s=1}^{\infty} f_s^2 - 32j^2 a_0 \right) = f_j \left( 6 \sum_{s=1}^{\infty} f_s^2 + \frac{2(2k^2 - 1)}{\gamma^2} + j^2 \left( \frac{1}{8} - 32a_0 \right) \right) = 0.
\]

System (7) has a nonzero solution if and only if

\[
\begin{cases}
\omega_1 = \frac{1}{256}, \\
\sum_{n=1}^{\infty} f_n = 1 - 2k^2 \frac{1}{3\gamma^2}.
\end{cases}
\]

We have obtained the value of \( \omega_1 \). The second equation of this system is equivalent to the following
equation in parameter $q$:

$$3 \sum_{n=1}^{\infty} \left( \frac{q^{n-(1/2)}}{1 + q^{n-1}} \right)^2 - \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{n-1}} \right)^2 + 2 \left( \sum_{n=1}^{\infty} \frac{q^{n-(1/2)}}{1 + q^{n-1}} \right)^2 = 0.$$  \hspace{1cm} (11)

This equation has the following solution on interval $(0,1)$:

$$q = 1.42142623201 \times 10^{-2} \pm 1 \times 10^{-13}.$$  

Thus the lemma is proved. \hfill \Box

Now it is easy to construct the required leading-order approximation for the function $\phi(x, \t)$:

$$\phi_0(x, \t) = A \{ \text{cn}(\alpha(x - \t), k) - \text{cn}(\alpha(x + \t), k) \}.$$  

For arbitrary $k \in (0, 1)$ this function is a real solution of Eq. (6). If $\alpha = 2K/\pi$, then the periods of $\phi_0(x, \t)$ in $x$ and in $\t$ are equal to $2\pi$. Using the Fourier-series expansion for the function cn$(z, k)$ (formula (8)), we obtain the following expansion for function $\phi_0(x, \t)$:

$$\phi_0(x, \t) = 2Ay_k \sum_{n=1}^{\infty} f_{2n-1} \sin((2n - 1)x) \sin((2n - 1)t).$$

If $q = 1.42142623201 \times 10^{-2} \pm 1 \times 10^{-13}$, then $q$ is a solution of Eq. (11) and the sequence $f$ is a real solution of system (7). All equations in system (7) are homogeneous, hence for these values of parameters, the sequence of the Fourier coefficients of the function $\phi_0(x, \t)$ also is a solution of system (7), with

$$\omega_0 = \frac{y^2}{64k}A^2 = 1.0983600974A^2.$$  

Thus we have proved that the function

$$\phi_0(x, \t) = A \{ \text{cn}(\alpha(x - \t), k) - \text{cn}(\alpha(x + \t), k) \},$$

with $k = 0.45107559881$ and $\alpha = 1.0576653982$ is such a standing wave solution of Eq. (5) that Eq. (6) has a periodic solution.

### 3.5. The first approximation

Now it is easy to find the periodic solution $\phi_1(x, \t)$. Let us designate the Fourier coefficients of the function $\phi_0^2(x, \t)$ as $D_{0j}$:

$$\phi_0^2(x, \t) = \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} D_{0j} \sin(nx) \sin(jt).$$

$$\phi_1(x, \t) = \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} D_{0j} \sin(nx) \sin(jt).$$
Eq. (6) gives the following result \((b_{nn}\text{ are arbitrary numbers})\):

\[
\psi_1(x, \tilde{t}) = \sum_{n=1}^{\infty} \sum_{j=1, j \neq n}^{\infty} D_{nj} \frac{\sin(nx) \sin(j \tilde{t})}{n^2 - j^2} + \sum_{n=1}^{\infty} b_{nn} \sin(nx) \sin(n \tilde{t}).
\]

It should be noted that the function \(\psi_1(x, \tilde{t})\) with arbitrary diagonal coefficients \(b_{nn}\) is a solution of Eq. (6) and that all off-diagonal coefficients of \(\psi_1(x, \tilde{t})\) are proportional to \(A^3\).

3.6. The second approximation

Let us consider the approximation to second order in \(\varepsilon\):

\[
\frac{\partial^2 \psi_2(x, \tilde{t})}{\partial x^2} - \frac{\partial^2 \psi_2(x, \tilde{t})}{\partial \tilde{t}^2} = 2\omega_1 \frac{\partial^2 \psi_1(x, \tilde{t})}{\partial \tilde{t}^2} + (2\omega_2 + \omega_1^2) \frac{\partial^2 \psi_0(x, \tilde{t})}{\partial \tilde{t}^2} + 3\psi_1(x, \tilde{t})\psi_0(x, \tilde{t}),
\]

(12)

If all diagonal coefficients of \(\psi_1(x, \tilde{t})\) are zeros: \(\forall n : b_{nn} = 0\), then \(\forall j, n : b_{jn} = -b_{nj}\), and the function \(\psi_1(x, \tilde{t})\psi_0(x, \tilde{t})\) has no diagonal harmonics. Hence, selecting \(\omega_2 = -\frac{1}{2}\omega_1^2\), we obtain a periodic solution to Eq. (12):

\[
\psi_2(x, \tilde{t}) = \sum_{n=1}^{\infty} \sum_{j=1, j \neq n}^{\infty} H_{nj} \frac{\sin(nx) \sin(j \tilde{t})}{n^2 - j^2} + \sum_{n=1}^{\infty} b_{nn} \sin(nx) \sin(n \tilde{t}),
\]

where

\[
H(x, \tilde{t}) = 2\omega_1 \frac{\partial^2 \psi_1(x, \tilde{t})}{\partial \tilde{t}^2} + 3\psi_1(x, \tilde{t})\psi_0(x, \tilde{t}) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} H_{nj} \sin(nx) \sin(j \tilde{t}).
\]

It should be noted that all diagonal coefficients \(b_{nn}\) can be found only from the next order equation.

4. Conclusions

Using massless \(\phi^4\) theory as an example, we have shown that a uniform expansion of solutions for quasi-linear Klein–Gordon equations can be constructed even in the principal resonant case. To construct the uniform expansion we have used the Poincare–Lindstedt method and the nontrivial zero approximation: the function

\[
\psi_0(x, t) = A(\{\text{cn}(\alpha(x - \omega t), k) - \text{cn}(\alpha(x + \omega t), k)\}),
\]

with \(k = 0.451075598811\) and \(\alpha = 1.0576653982\).

Thus, using the Jacobi elliptic function cn instead of the trigonometric function cos, we have put the principal resonance to zero and constructed, with accuracy \(O(\varepsilon^3)\), the doubly periodic solution in the standing wave form:

\[
\psi(x, \omega t) = \psi_0(x, \omega t) + \varepsilon \psi_1(x, \omega t) + \varepsilon^2 \psi_2(x, \omega t) + O(\varepsilon^3).
\]
with the frequency
\[
\omega = 1 + \frac{\gamma^2}{64k^2} A^2 x^2 - \frac{\gamma^4}{8192k^4} A^4 x^4 + O(\epsilon^3) \\
= 1 + 1.0983600974 A^2 x^2 - 0.6031974518 A^4 x^4 + O(\epsilon^3).
\]

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Appendix A

This program have been written in REDUCE 3.6.

The program constructs system (7): \( R_{jk} = 0 \) and finds a particular solution of this system. The function \( \psi_0(x, t) \) has been denoted as \( \text{phi}(xx, tt) \). This function is a solution of Eq. (5). We assume that the function \( \text{phi}(xx, tt) \) is real and includes only odd harmonics:

\[
\text{phi}(xx, tt) = \sum^{n}_{j=1} a(2j - 1) \sin((2j - 1)xx) \sin((2j - 1)tt).
\]

The number of unknowns (parameter \( n \)) can be selected arbitrary. For \( n = 10 \) this program gives result on computer with 16 Mbytes operating memory, for \( n = 20 \) this program gives result on computer with 128 Mbytes operating memory.

Using the procedure “fourier” we expand Eq. (6) in the Fourier series and obtain the list of equations “listequa”. The number of equations is \( 3n \). The unknowns \( a(j) \) are Fourier coefficients of some function, hence, we can assume that the sequence of the numbers \( c(j) = a(j)/a(1) \), satisfies the following condition: \( \forall j > 1: |c(j)| < 1 \).

At the first step we assume that for \( j > 3 \), all \( c(j) = 0 \). From the first equation of “listequa” we find that \( C_{omega} \) is a polynomial in \( c(3) \). We substitute this value of \( C_{omega} \) in the second equation of “listequa”. As result we obtain that this equation is a cubic equation in \( c(3) \). We solve this equation, using the standard procedure SOLVE. One of its solution must be real and we select this solution as value of \( c(3) \).

At the second step we assume that \( c(5) \) is not zero \{clear \( c(5) \)\}. From the first equation of “listequa” we obtain that \( C_{omega} \) is a polynomial in \( c(5) \), substitute this value of \( C_{omega} \) in the third equation of “listequa” and solve it, determining \( c(5) \).

At the third step we substitute new values of \( c(5) \) and \( C_{omega} \) in the second equation of “listequa”, so we calculate the variable “equa”. If “equa” is more than \( \delta = 10^{-11} \), then we resolve the second equation of “listequa” and repeat the second step. Alternatively, if the obtained value of “equa” is less than or equal to \( \delta = 10^{-12} \), then we assume that the three leading equations of “listequa” are solved and seek \( c(7) \) from the fourth equation of “listequa” and so on. Once the coefficient \( c(2n - 1) \) has been found the program finishes to work.
out `rjj.res';
in fourier$
operator a,c,phi$
depend phi(xx,tt)$

% Expand the unknown function phi(xx,tt), which is a solution of
% equation (5), and equation (6) in the Fourier series.
phi(xx,tt)=for j=1:n sum a(2j-1)*sin((2j-1)*xx)*sin((2j-1)*tt)$
equation1:=2*w1*df(phi(xx,tt),tt,2)+
fourier{fourier{phi(xx,tt)**3,xx},tt}$_$
listequa:={}$

% The frequency w_1 is proportional to a(1)^2.
w1:=C_omega*a(1)**2$_$
for j=2:n do a(2*j-1):=c(2*j-1)*a(1)$_$
equation1:=equation1$_$
% Construct the system (7): R_{jj}(c)=0.
for k=1:3*n do
  listequa:append(listequa,
    {16*df(equation1,sin((2*k-1)*xx),
     sin((2*k-1)*tt))/a(1)**3});
% Find a real solution of this system.
on rounded$
C_omega:=0$_$
% C_omega must be connected variable, because I want to clear it
% without message: `WARNING...'.
for k=2:n do
  for j=k:n do c(2*j-1):=0;
  clear C_omega, c(2*k-1)$_$
  firstequa:=first(listequa)$_$
  C_omega:=C_omega-firstequa/d(1,1,1,c(2*k-1));
  equa:part(listequa,k)$_$
% All equations, which we solve using the procedure SOLVE, are
% cubic equations, hence, they have at least one real root. We
% assume
% that the absolute value of one of real roots is less than unit.
% We solve the equation and select this real solution
% as value of c(2*k-1).
solve_equa:=solve(equa,c(2*k-1));
while solve_equa neq {} do
  c(2*k-1):=part(solve_equa,1,2)$_$
  if(c(2*k-1)< sub(i=-i,c(2*k-1)) and abs(c(2*k-1))<1)
    then solve_equa:={}$_$
  else solve_equa:=rest(solve_equa)$_$
  >>;
test:=0;
while(test=0) do
    test:=1;
clear C_omega;
firstequa:=first(listequa);
C_omega:=C_omega-firstequa/df(firstequa,C_omega);
for j:=2;k do
    equa:=part(listequa,j);
    if(abs(equa)>10**(-11)) then
        clear c(2*j-1);
        test:=0;
equa:=part(listequa,j);
solve_equa=solve(equa,c(2*j-1));
    while solve_equa neq {} do
        c(2*j-1):=part(solve_equa,1,2);
        if(c(2*j-1)=sub{i=-i,c(2*j-1)} and abs(c(2*j-1))<1) then
            solve_equa:={}
        else solve_equa:=rest(solve_equa);
    end;
end;

% Write the obtained result.
c(1):=1$
for j:=1:n do write 'c('',2*j-1,'')='', c(2*j-1);
for j:=2:n do write 'c('',2*j-3,'')/c('',2*j-1,'')='', c(2*j-3)/
c(2*j-1);
write 'C_omega=''', C_omega;
for j:=1:3*n do write 'R('',j,'',',',j,'')='',part(listequa,j);
quit;
end;

The procedure “fourier” expands polynomials of sin(x) and cos(x) in Fourier series.

procedure fourier(FF,X);
% This procedure constructs Fourier-series expansions for polynomials
% of sin(x) and cos(x).
begin
    scalar F;
    for all a, b such that df(a,x) neq 0 and df(b,x) neq 0 let
        cos(a)*cos(b)=(cos(a-b)+cos(a+b))/2,
\[
\sin(a) \cdot \sin(b) = \frac{\cos(a-b) - \cos(a+b)}{2},
\sin(a) \cdot \cos(b) = \frac{\sin(a-b) + \sin(a+b)}{2},
\sin(a) \cdot \sin(b) = \frac{(1 - \cos(2a))}{2},
\cos(a) \cdot \cos(b) = \frac{(1 + \cos(2a))}{2};
\]

\text{F} := \text{FF};
for all \(a, b\) such that \(\text{df}(a, x) \neq 0\) and \(\text{df}(b, x) \neq 0\)
clear \(\cos(a) \cdot \cos(b), \sin(a) \cdot \sin(b), \sin(a) \cdot \cos(b), \sin(a) \cdot \sin(b), \sin(a) \cdot \sin(b), \cos(a) \cdot \cos(b);\)
return \text{F};
end;

References